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UNDERWATER ACOUSTICS: A SPREADING LOSS EXPRESSION WHICH PERMITS THE USE OF OCEAN BOTTOM CONTOURS

by

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ABSTRACT. This report develops a general spreading loss expression and ray tracing procedure for use in sonar detection studies particularly where shallow water makes bottom topography a significant factor. Any analytically describable ocean bottom can be accommodated by these techniques.

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FOREWORD

Underwater acoustics is a field of great complexity about which little is clearly understood. The variability of the ocean medium and the presence of nonuniformities and gross anomalies present overwhelming obstacles to a neat mathematical description of underwater acoustical phenomena.

Ocean-bottom topography, which is one of the most significant environmental factors in shallow water areas, and its effect on underwater sound transmissions are examined in this report. The analytical techniques presented herein enable increased accuracy in computations of sonar-energy loss along the transmission path.

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INTRODUCTION

Acoustic detection studies often rely on ray tracing procedures which analytically model the transmission of sound in an underwater environment. These procedures include the calculation of spreading loss, which is the resultant decrease in sound level as the wave front spreads over a generally expanding area. However, most applications use a spreading loss expression which omits allowances for a non-level ocean bottom and may lead to serious inaccuracies in those shallow-water or long-range acoustic detection studies where bottom-reflected sound is of major significance. This report eliminates such trouble by developing a general spreading loss expression and ray tracing procedure which may be used with any analytically describable ocean bottom.

THE COORDINATE SYSTEM

A coordinate system using orthogonal axes X , Y , Z , and angles θ and ϕ as shown in Fig. 1 is used throughout the derivation.

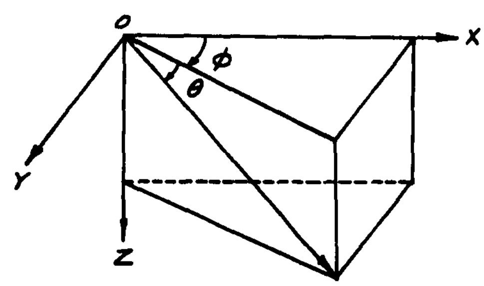

FIG. 1. The Coordinate System.

BASIC ASSUMPTIONS

1. The ocean medium is made up of a series of horizontal layers of thickness ΔZ , with each layer containing a constant velocity gradient $k = \Delta V / \Delta Z$, where ΔV is the difference in sound velocities at the layer boundaries.

2. Any analytically describable ocean bottom may be used.

3. The ocean surface is a horizontal plane.

Under the above assumptions, sound will travel from a source point P_o to a point P_n along a ray path which is contained in a series of vertical planes, with each vertical plane T_j containing the ray path between the jth and the j + 1st bottom reflections. $T_j = T_{j-1}$ if, and only if, the normal to the bottom at the jth reflection lies on vertical plane T_{j-1} . The course of the ray path on each vertical plane will be determined by refraction and surface reflection. Refraction is based on Snell's Law: $V_a / \cos \theta_a = V_b / \cos \theta_b$, where V_a, V_b , θ_a , and θ_b are the respective sound velocities and angles of inclination of the ray at any two depths, Z_a and Z_b , on that portion of the ray between successive boundary reflections.

The initial ray direction can be uniquely represented by two parameters, θ_o and ϕ_o , where θ_o is the angle the ray initially makes with the horizontal and ϕ_o is the angle between the vertical T_o and XZ planes. Given specific environmental parameters and a ray source point P_o , the coordinates (X, Y, Z) of any point P on a ray are functions of initial ray direction (θ_o , ϕ_o) and some third parameter such as travel time, ray path length, or horizontal distance covered, which fixes the location of the point on the ray.

THE SPREADING LOSS TERM

The spreading loss term $(SL)_n$ is the ratio of acoustic intensity I_n at point P_n to the acoustic intensity I_o at an index point a unit path length from source P_o . All acoustic energy is contained within a ray bundle as defined in the next paragraph, and a sound wave front at P_n creates a ray bundle cross-sectional area A_n. If energy is assumed evenly distributed over A_n , it follows that I_n is inversely proportional to A_n . Therefore, if A_o is a similar cross-sectional area at the index point,

$$(SL)_n = I_n / I_o = \left| A_o / A_n \right| \qquad (1)$$

RAY BUNDLE CONCEPT

Given a basic ray which leaves P_o with initial direction (θ_o, ϕ_o), a ray bundle is that volume which is bounded by the four ray paths leaving P_o with initial directions:

1. $(\theta_o + \Delta\theta_o, \phi_o + \Delta\phi_o)$
2. $(\theta_o + \Delta\theta_o, \phi_o - \Delta\phi_o)$
3. $(\theta_o - \Delta\theta_o, \phi_o + \Delta\phi_o)$
4. $(\theta_o - \Delta\theta_o, \phi_o - \Delta\phi_o)$

where $\Delta\theta_o$ and $\Delta\phi_o$ are infinitesimal.

Bounding rays 1, 2, 3, and 4 can be generated by considering two other rays, 5 and 6, whose respective initial directions are $(\theta_o + \Delta\theta_o, \phi_o)$ and $(\theta_o - \Delta\theta_o, \phi_o)$ as shown in Fig. 2. These two rays and the basic ray initially lie on the same vertical plane, T_o. By rotating T_o through the incremental angle $\Delta\phi_o$ about the vertical axis passing through P_o, rays 5 and 6 form rays 1 and 3, respectively, which initially lie on a common vertical plane, T_o'. By rotating T_o through the angle $-\Delta\phi_o$, rays 5 and 6 form rays 2 and 4, respectively, which initially lie on a common vertical plane, T_o''.

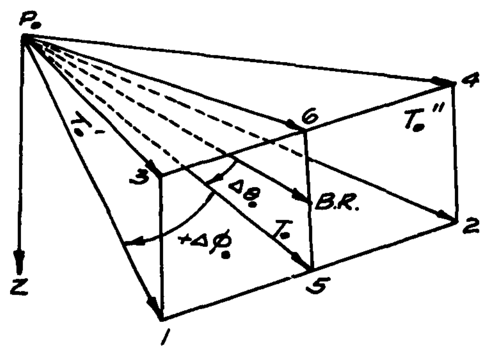

FIG. 2. The Ray Bundle.

CROSS-SECTIONAL AREA OF A RAY BUNDLE

The ray bundle cross-sectional area A_n is defined as the area of the sound wave front contained in the bundle at point P_n on the basic ray. A popular approximation of A_n is obtained from the product of equations 3B-38 and 3B-40 in Ref. (1), where it is assumed that the basic ray and the bounding rays are contained intirely in their respective initial vertical planes. However, since this assumption is not valid after the occurrence of a reflection from a sloping bottom, a more general approximation of A_n must be derived before any non-level ocean bottom is considered.

Let t_n be the time required for a sound wave to travel from source P_o to point P_n. A_n is the area of that surface which is described by the trace of point P_n as the initial ray direction varies from (θ_o, ϕ_o) to $(\theta_o \pm \Delta\theta_o, \phi_o \pm \Delta\phi_o)$, with t_n remaining constant. Since $\Delta\theta_o$ and $\Delta\phi_o$ are very small, A_n is a function of P_n, $\partial P_n/\partial\theta_o$, and $\partial P_n/\partial\phi_o$, where $\partial t_n/\partial\theta_o = 0$ and $\partial t_n/\partial\phi_o = 0$.

The first basic assumption indicates that the ocean medium has a velocity structure in which the velocity gradient is a piecewise constant function of depth. The coordinates of point P_n are (X_n, Y_n, Z_n) where Z_n, the depth of P_n, may be the depth of a velocity gradient

discontinuity. Therefore, it is convenient to approximate A_n from P_n, $\partial P_n/\partial \theta_o$ and $\partial P_n/\partial \phi_o$ with $\partial Z_n/\partial \theta_o = 0$ and $\partial Z_n/\partial \phi_o = 0$.

Consider the quadrilateral Q_n with vertices P_{1n} , P_{2n} , P_{3n}, and P_{4n} which are the respective points of intersection of bounding rays 1 through 4 with horizontal plane H_n passing through point P_n . The area of Q_n is obviously the horizontal cross-sectional area of the ray bundle at P_n . The desired cross-sectional area A_n is approximately equal to the area of the perpendicular projection of Q_n onto H_n^* , the basic ray's normal plane at P_n. If α_n is the angle between planes H_n and H_n^* ,

$$A_n = \cos \alpha_n \text{ (area of } Q_n). \tag{2}$$

An example of the perspective of planes H_n and H_n^* is given in Fig. 3. The ray direction at point P_n is (θ_n , ϕ_m), where θ_n is the ray's angle of inclination and ϕ_m is the angle between the vertical XZ plane and the vertical plane containing the ray path after the mth bottom reflection.

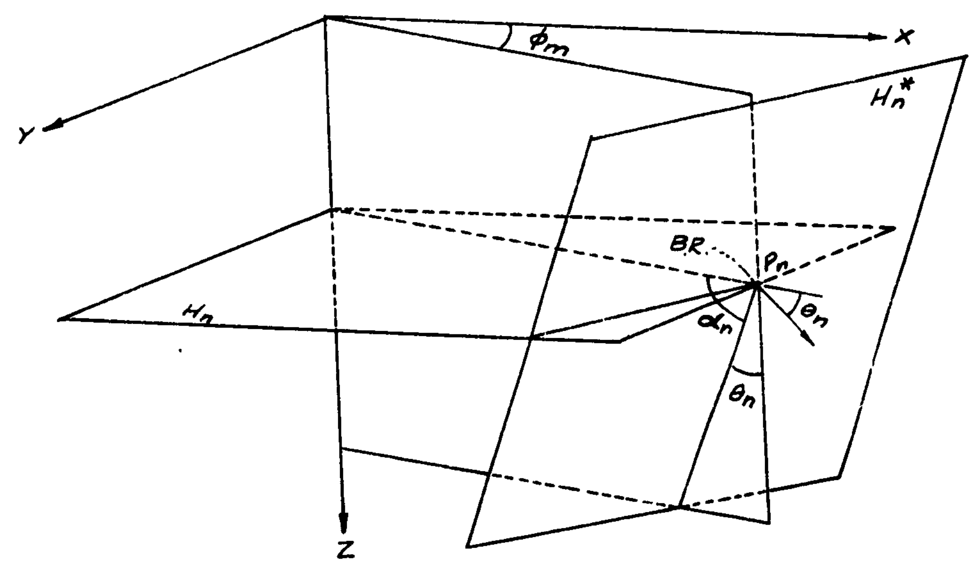

FIG. 3. Point P_n with Associated Planes.

It can be seen from Fig. 3 that $\alpha_n = 90^o - \theta_n$. Substituting this into equation (2),

$$A_n = \sin \theta_n \text{ (area of } Q_n). \tag{3}$$

The basic ray, defined with initial direction (θ_o , ϕ_o), intersects horizontal plane H_n at point P_n, which is at depth Z_n . Rays 5 and 6, previously defined with respective initial directions $(\theta_o + \Delta\theta_o , \phi_o)$ and $(\theta_o - \Delta\theta_o , \phi_o)$, will intersect H_n at points P_{5n} and P_{6n}, where

$$P_{5n} = P_n + (\partial P_n/\partial\theta_o)\,\Delta\theta_o \qquad P_{6n} = P_n - (\partial P_n/\partial\theta_o)\,\Delta\theta_o \tag{4}$$

with $\partial Z_n/\partial\theta_o = 0$.

Bounding rays 1 through 4, defined with respective initial directions $(\theta_o + \Delta\theta_o\ ,\ \phi_o + \Delta\phi_o)$, $(\theta_o + \Delta\theta_o\ ,\ \phi_o - \Delta\phi_o)$, $(\theta_o - \Delta\theta_o\ ,\phi_o + \Delta\phi_o)$, and $(\theta_o - \Delta\theta_o\ ,\ \phi_o - \Delta\phi_o)$, intersect H_n at points P_{1n} through P_{4n}, the vertices of quadrilateral Q_n. Therefore, using expression (4), the vertices of Q_n are

$$P_{1n} = P_{5n} + (\partial P_{5n}/\partial\phi_o)\Delta\phi_o \qquad P_{2n} = P_{5n} - (\partial P_{5n}/\partial\phi_o)\,\Delta\phi_o$$

$$P_{3n} = P_{6n} + (\partial P_{6n}/\partial\phi_o)\Delta\phi_o \qquad P_{4n} = P_{6n} - (\partial P_{6n}/\partial\phi_o)\,\Delta\phi_o \tag{5}$$

with $\partial Z_n/\partial\phi_o = 0$.

P_{1n} through P_{4n} can also be expressed in terms of two other rays, 7 and 8, whose respective initial directions are $(\theta_o\ ,\ \phi_o + \Delta\phi_o)$ and $(\theta_o\ ,\ \phi_o - \Delta\phi_o)$. Rays 7 and 8 will intersect H_n at points P_{7n} and P_{8n}, where

$$P_{7n} = P_n + (\partial P_n/\partial\phi_o)\,\Delta\phi_o \qquad P_{8n} = P_n - (\partial P_n/\partial\phi_o)\,\Delta\phi_o \tag{6}$$

with $\partial Z_n/\partial\phi_o = 0$.

Using expression (6), the vertices of Q_n are

$$P_{1n} = P_{7n} + (\partial P_{7n}/\partial\theta_o)\Delta\theta_o \qquad P_{2n} = P_{8n} + (\partial P_{8n}/\partial\theta_o)\Delta\theta_o$$

$$P_{3n} = P_{7n} - (\partial P_{7n}/\partial\theta_o)\Delta\theta_o \qquad P_{4n} = P_{8n} - (\partial P_{8n}/\partial\theta_o)\Delta\theta_o \tag{7}$$

with $\partial Z_n/\partial\theta_o = 0$.

Expressions (5) and (7) indicate that P_{5n}, P_{6n}, P_{7n}, and P_{8n} are midpoints of the sides of Q_n, and expressions (4) and (6) indicate that P_n is the point of intersection of lines joining opposite midpoints. Figure 4 is a diagram of Q_n with the related points described in expressions (4) through (7).

It is proved in the Appendix that the area of any quadrilateral equals eight times the area of any triangle whose vertices are the midpoints of two adjacent sides of the quadrilateral and the point of intersection of the two lines joining opposite midpoints. If $\bar{A}_n$ is the area of triangle $P_nP_{5n}P_{7n}$ in Fig. 4,

$$\text{Area of } Q_n = 8\bar{A}_n \tag{8}$$

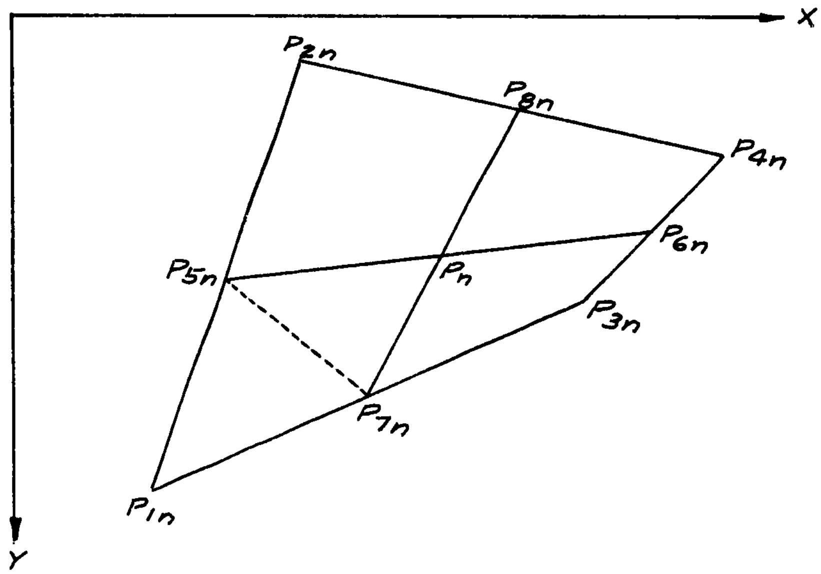

FIG. 4. Quadrilateral Q_n.

Combining equations (3) and (8),

$$A_n = 8\overline{A}_n \sin \theta_n \tag{9}$$

The coordinates of point P_n are (X_n , Y_n, Z_n). Therefore, from expressions (4) and (6),

$$\begin{aligned} &\text{Coordinates of } P_{5n} = (X_n + \Delta X_{5n}, Y_n + \Delta Y_{5n}, Z_n) \\ &\text{Coordinates of } P_{7n} = (X_n + \Delta X_{7n}, Y_n + \Delta Y_{7n}, Z_n) \end{aligned} \tag{10}$$

where

$$\begin{aligned} \Delta X_{5n} &= (\partial X_n/\partial \theta_o)\Delta \theta_o \qquad & \Delta Y_{5n} &= (\partial Y_n/\partial \theta_o)\Delta \theta_o \\ \Delta X_{7n} &= (\partial X_n/\partial \phi_o)\Delta \phi_o \qquad & \Delta Y_{7n} &= (\partial Y_n/\partial \phi_o)\Delta \phi_o \end{aligned} \tag{11}$$

The relative positions of points P_n , P_{5n} , and P_{7n} are shown in Fig. 5, where the coordinate system origin is at P_n.

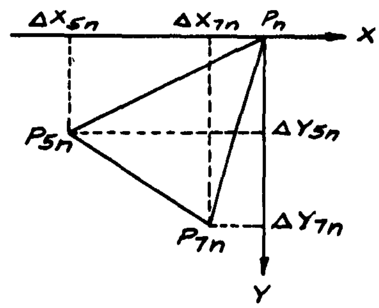

FIG. 5. Triangle $P_nP_{5n}P_{7n}$.

From Fig. 5, the area of triangle $P_nP_{5n}P_{7n}$ is

$$\bar{A}_n = -\left[\Delta X_{5n}\Delta Y_{7n} - \frac{\Delta X_{5n}\Delta Y_{5n}}{2} - \frac{\Delta X_{7n}\Delta Y_{7n}}{2} - \frac{(\Delta X_{5n} - \Delta X_{7n})(\Delta Y_{7n} - \Delta Y_{5n})}{2}\right] \quad (12)$$

$$= \frac{\Delta X_{7n}\Delta Y_{5n} - \Delta X_{5n}\Delta Y_{7n}}{2}$$

Combining equations (9) and (12),

$$A_n = 4 \sin \theta_n \,(\Delta X_{7n}\Delta Y_{5n} - \Delta X_{5n}\Delta Y_{7n}) \quad (13)$$

Substituting expression (11) into equation (13), the desired expression for the ray bundle cross-sectional area at point P_n is

$$A_n = 4\Delta\theta_o\Delta\phi_o \sin\theta_n \left(\frac{\partial X_n}{\partial\phi_o}\frac{\partial Y_n}{\partial\theta_o} - \frac{\partial X_n}{\partial\theta_o}\frac{\partial Y_n}{\partial\phi_o}\right) \quad (14)$$

There is a certain advantage in retaining the algebraic sign of A_n even though it has no significance as applied in equation (1). If A_n and A_{n+1} have opposite signs, the ray bundle has passed through a focussing point somewhere between points P_n and P_{n+1}.

RAY BUNDLE CROSS-SECTIONAL AREA AT THE INDEX POINT

A_o is the sound wave front area in the ray bundle at the index point unit path length from the ray source. Assuming that the ray path is a straight line over the first unit length, A_o is approximately the area of that surface which is generated by the movement of the index point as the initial ray direction varies from (θ_o, ϕ_o) to $(\theta_o \pm \Delta\theta_o, \phi_o \pm \Delta\phi_o)$, as described in Fig. 6.

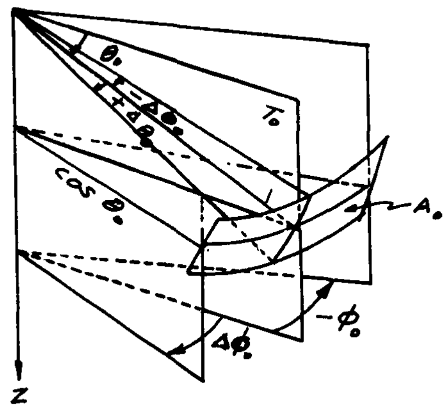

FIG. 6. Area A_o.

As θ_o is varied from $\theta_o - \Delta\theta_o$ to $\theta_o + \Delta\theta_o$, the index point describes an arc of length $2\Delta\theta_o$, lying on vertical plane T_o. As ϕ_o is varied from $\phi_o - \Delta\phi_o$ to $\phi_o + \Delta\phi_o$, the index point describes an arc of length $2\Delta\phi_o \cos\theta_o$, lying on a horizontal plane. Therefore, the ray bundle cross-sectional area at the index point is

$$A_o = 4\Delta\theta_o\Delta\phi_o \cos\theta_o \tag{15}$$

A GENERAL EXPRESSION FOR THE SPREADING LOSS TERM

Combining equations (1), (14), and (15), the spreading loss term at point P_n is

$$(SL)_n = \left| \frac{4\Delta\theta_o\Delta\phi_o \cos\theta_o}{4\Delta\theta_o\Delta\phi_o \sin\theta_n \left(\frac{\partial X_n}{\partial\phi_o}\frac{\partial Y_n}{\partial\theta_o} - \frac{\partial X_n}{\partial\theta_o}\frac{\partial Y_n}{\partial\phi_o} \right)} \right|$$

$$(SL)_n = \left| \frac{\cos\theta_o}{\sin\theta_n} \left(\frac{\partial X_n}{\partial\theta_o}\frac{\partial Y_n}{\partial\phi_o} - \frac{\partial X_n}{\partial\phi_o}\frac{\partial Y_n}{\partial\theta_o} \right)^{-1} \right| \tag{16}$$

AN OCEAN BOTTOM BOUNDARY CONDITION

Ray bundle cross-sectional area A_n was derived from points P_n, P_{5n}, and P_{7n}, the respective intersections of the basic ray and rays 5 and 7 with horizontal plane H_n. However, when P_n is the intersection of the basic ray and a non-level ocean bottom, points P_{5n} and P_{7n} are

not uniquely defined since rays 5 and 7 will intersect plane H_n twice. This difficulty is circumvented by assuming that points P_{5n} and P_{7n} apply to rays 5 and 7 before reflection and by introducing two new points, P'_{5n} and P'_{7n}, which apply to the rays after reflection. Two-dimensional examples of the situation are shown in Fig. 7 and 8.

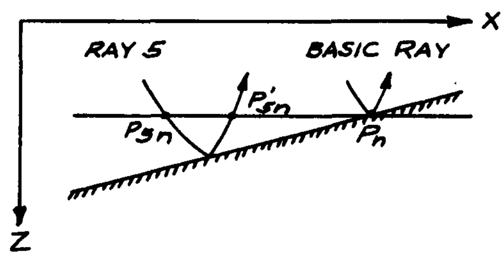

FIG. 7. Up-Slope Reflection.

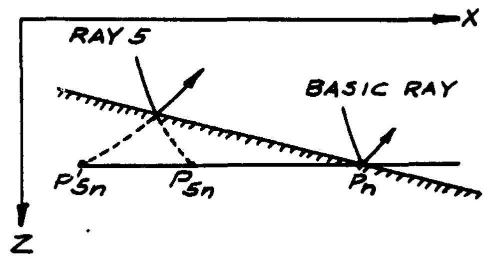

FIG. 8. Down-Slope Reflection.

The coordinates of point P_n are (X_n, Y_n, Z_n). When P_n is a bottom reflection point, the before- and after-reflection rates of change of X_n and Y_n shall be respectively represented by unprimed and primed partial derivatives. Expressions (10) through (14) will therefore determine the before-reflection ray bundle cross-sectional area, with the after-reflection area similarly found from points P_n, P'_{5n}, and P'_{7n} where

$$\begin{aligned} &\text{Coordinates of } P'_{5n} = (X_n + \Delta X'_{5n},\ Y_n + \Delta Y'_{5n},\ Z_n) \\ &\text{Coordinates of } P'_{7n} = (X_n + \Delta X'_{7n},\ Y_n + \Delta Y'_{7n},\ Z_n) \end{aligned} \tag{17}$$

with

$$\begin{aligned} \Delta X'_{5n} &= (\partial X'_n/\partial \theta_o)\Delta\theta_o \qquad & \Delta Y'_{5n} &= (\partial Y'_n/\partial \theta_o)\Delta\theta_o \\ \Delta X'_{7n} &= (\partial X'_n/\partial \phi_o)\Delta\phi_o \qquad & \Delta Y'_{7n} &= (\partial Y'_n/\partial \phi_o)\Delta\phi_o \end{aligned} \tag{18}$$

It remains to relate the above after-reflection terms to the before-reflection terms in expressions (10) and (11), thereby obtaining the primed partial derivatives of X_n and Y_n which may be used in equation (14) to calculate the after-reflection cross-sectional area.

For the moment, let P_n be the basic ray's <u>mth</u> bottom reflection point. Let the ocean bottom tangent plane at this reflection point be E_m, defined by P_n and two angles of inclination, ψ_m and σ_m, which are the angles plane E_m makes with the horizontal in the respective XZ and YZ planes. Let the bottom reflection entrant and emergent basic ray directions be (θ_n, ϕ_{m-1}) and (θ'_n, ϕ_m), where θ is the ray's angle of inclination and ϕ is the angle between the vertical XZ plane and the vertical plane containing the ray path. The respective initial directions of the basic ray and rays 5 and 7 were previously defined as (θ_o, ϕ_o), ($\theta_o + \Delta\theta_o$, ϕ_o), and (θ_o, $\phi_o + \Delta\phi_o$). Since $\Delta\theta_o$ and

$\Delta\phi_o$ are infinitesimal, adequate approximations of desired points P'_{5n} and P'_{7n} are obtained by assuming that rays 5 and 7 travel in straight lines with basic ray entrant direction (θ_n, ϕ_{m-1}) from respective points P_{5n} and P_{7n} to plane E_m , reflect, and travel in straight lines with basic ray emergent direction (θ'_n, ϕ_m) from E_m to points P'_{5n} and P'_{7n} . An example of the relationship of point P'_{5n} to P_n and P_{5n} is shown in Fig. 9, where plane E_m is described by point P_n , a negative ψ_m and a positive σ_m. (The relationship of P'_{7n} to P_n and P_{7n} will not be discussed further, since it is similar to that for P'_{5n} .)

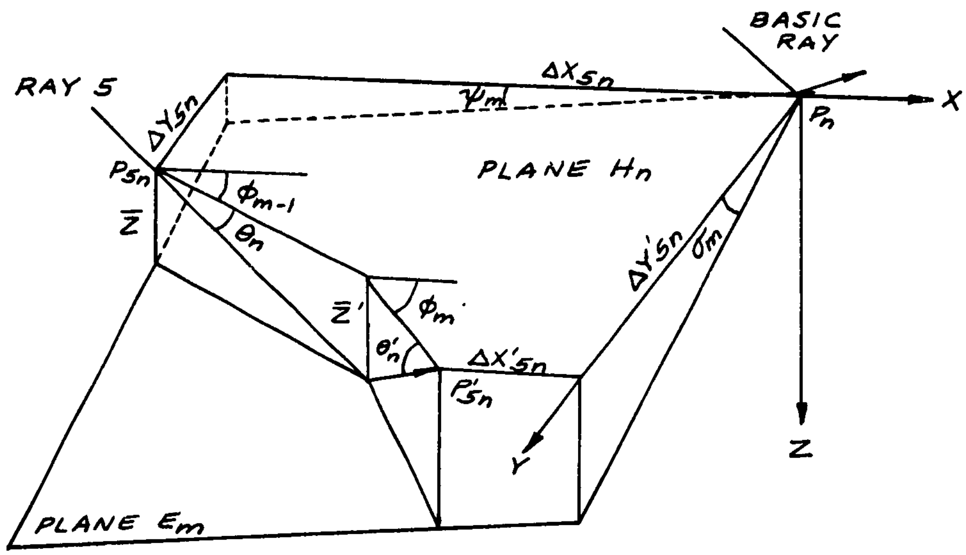

FIG. 9. Relative Positions of Points P_n , P_{5n} , and P'_{5n}.

From Fig. 9, the vertical distance between point P_{5n} and plane E_m is

$$\bar{Z} = \Delta X_{5n} \tan \psi_m + \Delta Y_{5n} \tan \sigma_m . \tag{19}$$

A top view of Fig. 9 is shown in Fig. 10 where, noting that ΔX_{5n} and $\Delta X'_{5n}$ are negative,

$$\begin{aligned} \Delta X'_{5n} &= \Delta X_{5n} + \bar{X} + \bar{X}' \\ \Delta Y'_{5n} &= \Delta Y_{5n} + \bar{Y} + \bar{Y}' \end{aligned} \tag{20}$$

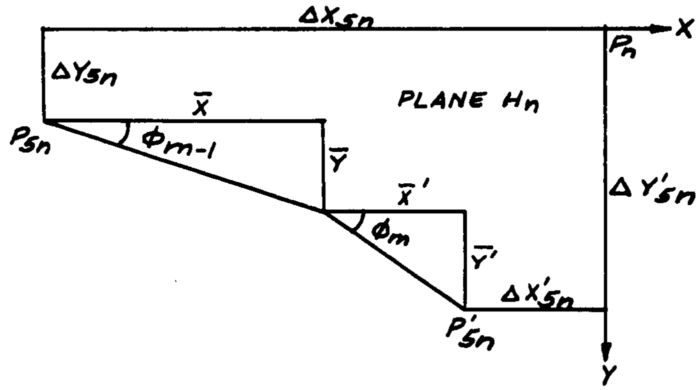

FIG. 10.

A closeup of ray 5 as it travels from point P_{5n} to P'_{5n} is shown in Fig. 11, where it can be seen that

$$\bar{X} = M \cos \phi_{m-1} = \frac{\bar{Z}'}{\tan \theta_n} \cos \phi_{m-1}$$

$$\bar{X}' = M' \cos \phi_m = - \frac{\bar{Z}'}{\tan \theta'_n} \cos \phi_m \tag{21}$$

$$\bar{Y} = M \sin \phi_{m-1} = \frac{\bar{Z}'}{\tan \theta_n} \sin \phi_{m-1}$$

$$\bar{Y}' = M' \sin \phi_m = - \frac{\bar{Z}'}{\tan \theta'_n} \sin \phi_m$$

Substituting expression (21) into (20),

$$\Delta X'_{5n} = \Delta X_{5n} + \bar{Z}' \frac{(\cos \phi_{m-1} \tan \theta'_n - \cos \phi_m \tan \theta_n)}{\tan \theta_n \tan \theta'_n}$$

$$\Delta Y'_{5n} = \Delta Y_{5n} + \bar{Z}' \frac{(\sin \phi_{m-1} \tan \theta'_n - \sin \phi_m \tan \theta_n)}{\tan \theta_n \tan \theta'_n} \tag{22}$$

Further inspection of Fig. 11 yields

$$\bar{Z}' = \bar{Z} + \bar{X} \tan \psi_m + \bar{Y} \tan \sigma_m$$

$$= \bar{Z} + \bar{X}(\tan \psi_m + \tan \phi_{m-1} \tan \sigma_m) \tag{23}$$

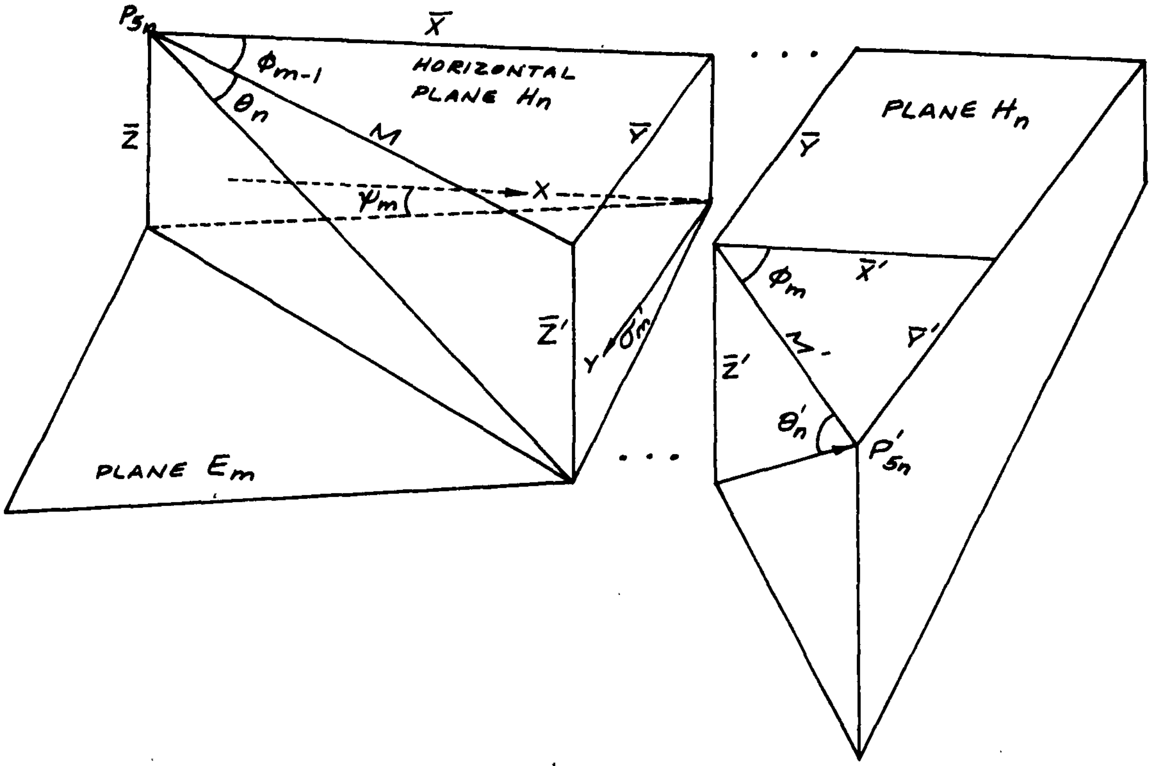

FIG. 11. Ray 5 Broken Closeup.

Combining equation (23) and the first equation in expression (21),

$$\bar{Z}' = \bar{Z} + \frac{\bar{Z}'}{\tan \theta_n} \cos \phi_{m-1} (\tan \psi_m + \tan \phi_{m-1} \tan \sigma_m)$$

$$= \bar{Z} + \frac{\bar{Z}'}{\tan \theta_n} (\cos \phi_{m-1} \tan \psi_m + \sin \phi_{m-1} \tan \sigma_m)$$

$$\bar{Z}' = \frac{\bar{Z} \tan \theta_n}{\tan \theta_n - \cos \phi_{m-1} \tan \psi_m - \sin \phi_{m-1} \tan \sigma_m}$$

Substituting the above into expression (22),

$$\Delta X'_{5n} = \Delta X_{5n} + \frac{\bar{Z}(\cos \phi_{m-1} \tan \theta'_n - \cos \phi_m \tan \theta_n)}{\tan \theta'_n (\tan \theta_n - \cos \phi_{m-1} \tan \psi_m - \sin \phi_{m-1} \tan \sigma_m)}$$

$$= \Delta X_{5n} + C_{m1} \bar{Z} \tag{24}$$

$$\Delta Y'_{5n} = \Delta Y_{5n} + C_{m2} \bar{Z}$$

where

$$C_{m1} = \frac{\cos \phi_{m-1} \tan \theta'_n - \cos \phi_m \tan \theta_n}{\tan \theta'_n (\tan \theta_n - \cos \phi_{m-1} \tan \psi_m - \sin \phi_{m-1} \tan \sigma_m)}$$

$$C_{m2} = \frac{\sin \phi_{m-1} \tan \theta'_n - \sin \phi_m \tan \theta_n}{\tan \theta'_n (\tan \theta_n - \cos \phi_{m-1} \tan \psi_m - \sin \phi_{m-1} \tan \sigma_m)} \tag{25}$$

Substituting equation (19) into expression (24),

$$\Delta X'_{5n} = \Delta X_{5n} + C_{m1} (\Delta X_{5n} \tan \psi_m + \Delta Y_{5n} \tan \sigma_m)$$

$$\Delta Y'_{5n} = \Delta Y_{5n} + C_{m2} (\Delta X_{5n} \tan \psi_m + \Delta Y_{5n} \tan \sigma_m)$$

Substituting pertinent equations from expressions (11) and (18) into the above, the desired ocean bottom boundary condition compensation for ray 5 is

$$\frac{\partial X'_n}{\partial \theta_o} = \frac{\partial X_n}{\partial \theta_o} + C_{m1} \left(\frac{\partial X_n}{\partial \theta_o} \tan \psi_m + \frac{\partial Y_n}{\partial \theta_o} \tan \sigma_m \right)$$

$$\frac{\partial Y'_n}{\partial \theta_o} = \frac{\partial Y_n}{\partial \theta_o} + C_{m2} \left(\frac{\partial X_n}{\partial \theta_o} \tan \psi_m + \frac{\partial Y_n}{\partial \theta_o} \tan \sigma_m \right) \tag{26}$$

Expression (26) displays a relationship of terms obtained from points P_n, P_{5n}, and P'_{5n}. Since the discussion on page 9 indicates that rays 5 and 7 are assumed to follow parallel courses (both with basic ray direction) in the space immediately adjacent to the basic ray's bottom reflection, expressions (10), (11), (17), and (18) are used to acquire the similar relationship of corresponding terms from points P_n, P_{7n}, and P'_{7n}:

$$\frac{\partial X_n'}{\partial \phi_o} = \frac{\partial X_n}{\partial \phi_o} + C_{m1}\left(\frac{\partial X_n}{\partial \phi_o}\tan\psi_m + \frac{\partial Y_n}{\partial \phi_o}\tan\sigma_m\right)$$
$$\frac{\partial Y_n'}{\partial \phi_o} = \frac{\partial Y_n}{\partial \phi_o} + C_{m2}\left(\frac{\partial X_n}{\partial \phi_o}\tan\psi_m + \frac{\partial Y_n}{\partial \phi_o}\tan\sigma_m\right) \tag{27}$$

In order to uniquely identify bottom reflection points, let n = Bm at the mth bottom reflection. Then P_{Bm} is the point of reflection, with $(\theta_{Bm}, \phi_{m-1})$ and (θ_{Bm}', ϕ_m) the bottom reflection entrant and emergent basic ray directions. The ocean bottom tangent plane at this reflection point is E_m, defined by point P_{Bm} and two angles of inclination, ψ_m and σ_m, which are the angles plane E_m makes with the horizontal in the respective XZ and YZ planes. The coordinates of point P_{Bm} are (X_{Bm}, Y_{Bm}, Z_{Bm}). Given $\partial X_{Bm}/\partial\theta_o, \partial Y_{Bm}/\partial\theta_o$, and $\partial X_{Bm}/\partial\phi_o$, and $\partial Y_{Bm}/\partial\phi_o$ immediately before reflection, expressions (25) through (27) yield the following corresponding terms immediately after reflection:

$$\frac{\partial X_{Bm}'}{\partial \theta_o} = \frac{\partial X_{Bm}}{\partial \theta_o} + C_{m1}\left(\frac{\partial X_{Bm}}{\partial \theta_o}\tan\psi_m + \frac{\partial Y_{Bm}}{\partial \theta_o}\tan\sigma_m\right)$$
$$\frac{\partial Y_{Bm}'}{\partial \theta_o} = \frac{\partial Y_{Bm}}{\partial \theta_o} + C_{m2}\left(\frac{\partial X_{Bm}}{\partial \theta_o}\tan\psi_m + \frac{\partial Y_{Bm}}{\partial \theta_o}\tan\sigma_m\right)$$
$$\frac{\partial X_{Bm}'}{\partial \phi_o} = \frac{\partial X_{Bm}}{\partial \phi_o} + C_{m1}\left(\frac{\partial X_{Bm}}{\partial \phi_o}\tan\psi_m + \frac{\partial Y_{Bm}}{\partial \phi_o}\tan\sigma_m\right) \tag{28}$$
$$\frac{\partial Y_{Bm}'}{\partial \phi_o} = \frac{\partial Y_{Bm}}{\partial \phi_o} + C_{m2}\left(\frac{\partial X_{Bm}}{\partial \phi_o}\tan\psi_m + \frac{\partial Y_{Bm}}{\partial \phi_o}\tan\sigma_m\right)$$

where

$$C_{m1} = \frac{\cos\phi_{m-1}\tan\theta_{Bm}' - \cos\phi_m\tan\theta_{Bm}}{\tan\theta_{Bm}'\left(\tan\theta_{Bm} - \cos\phi_{m-1}\tan\psi_m - \sin\phi_{m-1}\tan\sigma_m\right)}$$

$$C_{m2} = \frac{\sin\phi_{m-1}\tan\theta_{Bm}' - \sin\phi_m\tan\theta_{Bm}}{\tan\theta_{Bm}'\left(\tan\theta_{Bm} - \cos\phi_{m-1}\tan\psi_m - \sin\phi_{m-1}\tan\sigma_m\right)}$$

Let $(SL)_{Bm}$ and $(SL)_{Bm}'$ be the respective spreading loss terms immediately before and after the mth bottom reflection. Then, from equation (16),

$$(SL)_{Bm} = \left| \frac{\cos \theta_o}{\sin \theta_{Bm}} \left(\frac{\partial X_{Bm}}{\partial \theta_o} \frac{\partial Y_{Bm}}{\partial \phi_o} - \frac{\partial X_{Bm}}{\partial \phi_o} \frac{\partial Y_{Bm}}{\partial \theta_o} \right)^{-1} \right|$$

$$(SL)'_{Bm} = \left| \frac{\cos \theta_o}{\sin \theta'_{Bm}} \left(\frac{\partial X'_{Bm}}{\partial \theta_o} \frac{\partial Y'_{Bm}}{\partial \phi_o} - \frac{\partial X'_{Bm}}{\partial \phi_o} \frac{\partial Y'_{Bm}}{\partial \theta_o} \right)^{-1} \right| \qquad (29)$$

where the primed partial derivatives are obtained from expression (28).

CALCULATION OF SPECULARLY REFLECTED RAY DIRECTION

The mth bottom reflection entrant and emergent ray directions are $(\theta_{Bm}, \phi_{m-1})$ and (θ'_{Bm}, ϕ_m), where θ is the ray's angle of inclination and ϕ is the angle between the vertical XZ plane and the vertical plane containing the ray path. It is desired to express θ'_{Bm} and ϕ_m in terms of θ_{Bm} and ϕ_{m-1}.

Let V_m and V'_m be the respective mth bottom reflection entrant and emergent ray tangent unit vectors. Then, using Fig. 1, the direction cosines of V_m and V'_m are

$$V_m = (\cos \theta_{Bm} \cos \phi_{m-1}, \cos \theta_{Bm} \sin \phi_{m-1}, \sin \theta_{Bm})$$

$$V'_m = (\cos \theta'_{Bm} \cos \phi_m, \cos \theta'_{Bm} \sin \phi_m, \sin \theta'_{Bm}) \qquad (30)$$

Let N_m be a unit vector which is normal to the ocean bottom at the reflection point. N_m is also normal to E_m, the bottom tangent plane defined in the preceding section by the point of reflection and two angles of inclination, ψ_m and σ_m. Therefore, the direction cosines of N_m can be expressed in terms of ψ_m and σ_m. Figure 12 is an example of bottom normal N_m, with bottom tangent plane E_m described by a negative ψ_m and a positive σ_m. Noting from Fig. 12 that the direction cosine ratios of N_m are $(\tan \psi_m, \tan \sigma_m, -1)$, it follows that the direction cosines of N_m are

$$N_m = \left(\frac{\tan \psi_m}{\sqrt{C}}, \frac{\tan \sigma_m}{\sqrt{C}}, \frac{-1}{\sqrt{C}} \right) \qquad (31)$$

where

$$C = \tan^2 \psi_m + \tan^2 \sigma_m + 1$$

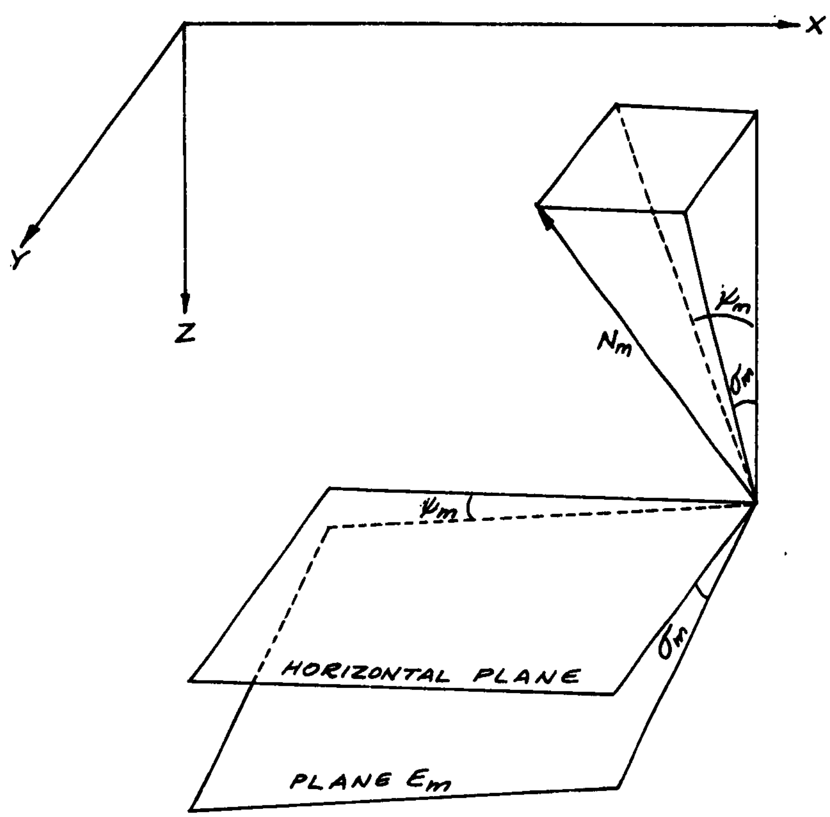

FIG. 12. Bottom Normal N_m.

Specular reflection imposes two conditions on the entrant and emergent ray tangent vectors, V_m and V_m', and the bottom normal, N_m. These conditions are

$$V_m' = aV_m + bN_m \qquad (V_m,\ V_m',\ \text{and } N_m \text{ are coplanar.}) \tag{32}$$

$$V_m' \cdot N_m = -V_m \cdot N_m \qquad (N_m \text{ bisects angle between } -V_m \text{ and } V_m'.) \tag{33}$$

Since the above vectors were defined with unit lengths,

$$V_m \cdot V_m = V_m' \cdot V_m' = N_m \cdot N_m = 1 \tag{34}$$

It remains to determine from equations (32), (33), and (34) a solution for V_m' in terms of V_m and N_m, which may be used with direction cosine expressions (30) and (31) to solve for θ_{Bm}' and ϕ_m. From equation (32),

$$V_m' \cdot (V_m + V_m') = aV_m \cdot (V_m + V_m') + bN_m \cdot (V_m + V_m')$$

$$V_m' \cdot V_m + V_m' \cdot V_m' = a(V_m \cdot V_m + V_m \cdot V_m') + b(N_m \cdot V_m + N_m \cdot V_m')$$

Substituting equations (33) and (34) into the above,

$$V_m' \cdot V_m + 1 = a(1 + V_m \cdot V_m')$$

$$a = 1$$

Equation (32) now becomes

$$V_m' = V_m + bN_m$$

$$V_m' \cdot N_m = V_m \cdot N_m + bN_m \cdot N_m \tag{35}$$

From equations (33), (34), and the above,

$$-V_m \cdot N_m = V_m \cdot N_m + b$$

$$b = -2V_m \cdot N_m$$

Substituting the above into equation (35), the desired solution for V_m' is

$$V_m' = V_m - 2(V_m \cdot N_m)N_m \tag{36}$$

Turning to the application of equation (36), note that the dot product of two unit vectors is equal to the sum of the products of their corresponding direction cosines. Therefore, from expressions (30) and (31),

$$V_m \cdot N_m = \cos \theta_{Bm} \cos \phi_{m-1} \frac{\tan \psi_m}{\sqrt{C}} + \cos \theta_{Bm} \sin \phi_{m-1} \frac{\tan \sigma_m}{\sqrt{C}} - \frac{\sin \theta_{Bm}}{\sqrt{C}}$$

$$= \sqrt{C} \cos \theta_{Bm} \left(\frac{\cos \phi_{m-1} \tan \psi_m + \sin \phi_{m-1} \tan \sigma_m - \tan \theta_{Bm}}{C} \right)$$

$$V_m \cdot N_m = C_{m3} \sqrt{C} \cos \theta_{Bm} \tag{37}$$

where

$$C_{m3} = \frac{\cos \phi_{m-1} \tan \psi_m + \sin \phi_{m-1} \tan \sigma_m - \tan \theta_{Bm}}{\tan^2 \psi_m + \tan^2 \sigma_m + 1}$$

Using expressions (30), (31), and (37) with (36), the direction cosines of V_m' are

$$\cos \theta'_{Bm} \cos \phi_m = \cos \theta_{Bm} \cos \phi_{m-1} - 2(C_{m3} \sqrt{C} \cos \theta_{Bm}) \frac{\tan \psi_m}{\sqrt{C}}$$

$$= \cos \theta_{Bm} (\cos \phi_{m-1} - 2C_{m3} \tan \psi_m) \qquad (38)$$

$$\cos \theta'_{Bm} \sin \phi_m = \cos \theta_{Bm} (\sin \phi_{m-1} - 2C_{m3} \tan \sigma_m) \qquad (38a)$$

$$\sin \theta'_{Bm} = \sin \theta_{Bm} + 2C_{m3} \cos \theta_{Bm}$$

The above equations immediately lend themselves to the following relationships between the mth bottom reflection entrant and emergent ray directions, $(\theta_{Bm}, \phi_{m-1})$ and (θ'_{Bm}, ϕ_m). .

$$\sin \theta'_{Bm} = \sin \theta_{Bm} + 2C_{m3} \cos \theta_{Bm} \qquad (39)$$

$$\tan \phi_m = \frac{\sin \phi_{m-1} - 2C_{m3} \tan \sigma_m}{\cos \phi_{m-1} - 2C_{m3} \tan \psi_m} \qquad (40)$$

with

$$C_{m3} = \frac{\cos \phi_{m-1} \tan \psi_m + \sin \phi_{m-1} \tan \sigma_m - \tan \theta_{Bm}}{\tan^2 \psi_m + \tan^2 \sigma_m + 1} \qquad (41)$$

EVALUATION OF THE GENERAL EXPRESSION FOR THE SPREADING LOSS TERM

Restating equation (16), the general expression for the spreading loss term at any point P_n on the ray path is

$$(SL)_n = \left| \frac{\cos \theta_o}{\sin \theta_n} \left(\frac{\partial X_n}{\partial \theta_o} \frac{\partial Y_n}{\partial \phi_o} - \frac{\partial X_n}{\partial \phi_o} \frac{\partial Y_n}{\partial \theta_o} \right)^{-1} \right| \qquad (42)$$

It remains to derive the stated partial derivatives of X_n and Y_n.

Point P_n has coordinates (X_n, Y_n, Z_n) which are functions of initial ray direction, (θ_o, ϕ_o), and some third parameter fixing the location of the point on the ray. Let this third parameter be S_n, the total horizontal distance covered by the ray from source P_o to point P_n. It was noted after the basic assumptions that a sound ray path will be contained in a series of vertical planes, with each plane containing the path between successive bottom reflections. A top view of a ray path with m bottom reflections is shown in Fig. 13, where ΔS_n is the horizontal distance covered as the ray travels from the mth bottom reflection to any point P_n before another bottom reflection, and ϕ is the angle between the vertical XZ plane and the vertical plane containing the ray path.

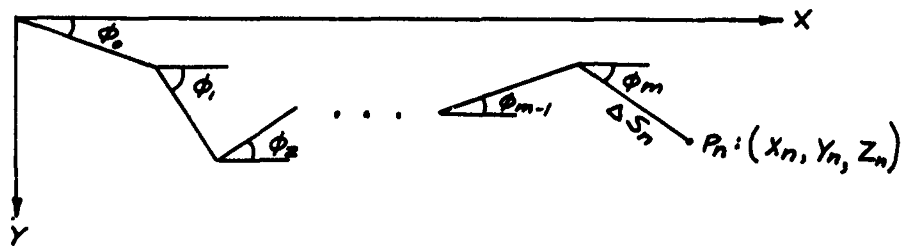

FIG. 13. Horizontal Distance.

The coordinates of the ray's mth bottom reflection point are (X_{Bm}, Y_{Bm}, Z_{Bm}). Therefore, from Fig. 13,

$$
\begin{aligned}
X_n &= X_{Bm} + \Delta S_n \cos \phi_m \\
Y_n &= Y_{Bm} + \Delta S_n \sin \phi_m
\end{aligned}
\tag{43}
$$

The boundary condition discussion preceding expression (28) used unprimed and primed partial derivatives to respectively denote before- and after-reflection rates of change. Bearing this in mind, expression (43) yields

$$
\begin{aligned}
\frac{\partial X_n}{\partial \theta_o} &= \frac{\partial X'_{Bm}}{\partial \theta_o} - \Delta S_n \sin \phi_m \frac{\partial \phi_m}{\partial \theta_o} + \cos \phi_m \frac{\partial \Delta S_n}{\partial \theta_o} \\
\frac{\partial Y_n}{\partial \theta_o} &= \frac{\partial Y'_{Bm}}{\partial \theta_o} + \Delta S_n \cos \phi_m \frac{\partial \phi_m}{\partial \theta_o} + \sin \phi_m \frac{\partial \Delta S_n}{\partial \theta_o} \\
\frac{\partial X_n}{\partial \phi_o} &= \frac{\partial X'_{Bm}}{\partial \phi_o} - \Delta S_n \sin \phi_m \frac{\partial \phi_m}{\partial \phi_o} + \cos \phi_m \frac{\partial \Delta S_n}{\partial \phi_o} \\
\frac{\partial Y_n}{\partial \phi_o} &= \frac{\partial Y'_{Bm}}{\partial \phi_o} + \Delta S_n \cos \phi_m \frac{\partial \phi_m}{\partial \phi_o} + \sin \phi_m \frac{\partial \Delta S_n}{\partial \phi_o}
\end{aligned}
\tag{44}
$$

where the primed partial derivatives are found from $\partial X_{Bm}/\partial \theta_o$, $\partial Y_{Bm}/\partial \theta_o$, $\partial X_{Bm}/\partial \phi_o$, and $\partial Y_{Bm}/\partial \phi_o$, using expression (28).

A general ray tracing procedure now begins to take shape. Assuming that expressions for ΔS_n, $\partial \Delta S_n/\partial \theta_o$, $\partial \Delta S_n/\partial \phi_o$, $\partial \phi_m/\partial \theta_o$, and $\partial \phi_m/\partial \phi_o$ are available, the spreading loss term at each desired point on a ray is obtained from expressions (42) and (44) where, before the first bottom reflection,

$$m = 0 \qquad \phi_m = \phi_o \qquad \partial\phi_m/\partial\phi_o = 1 \qquad Bm = 0$$

$$\partial X'_{Bm}/\partial\theta_o = 0 \qquad \partial Y'_{Bm}/\partial\theta_o = 0 \qquad \partial X'_{Bm}/\partial\phi_o = 0 \qquad \partial Y'_{Bm}/\partial\phi_o = 0$$

$$\partial\phi_m/\partial\theta_o = 0 \tag{45}$$

(since ϕ_o and θ_o are independent initial parameters.)

When the ray reaches the mth bottom reflection point, $\theta_n = \theta_{Bm}$,

$$\partial X_n/\partial\theta_o = \partial X_{Bm}/\partial\theta_o \qquad \partial Y_n/\partial\theta_o = \partial Y_{Bm}/\partial\theta_o$$
$$\partial X_n/\partial\phi_o = \partial X_{Bm}/\partial\phi_o \qquad \partial Y_n/\partial\phi_o = \partial Y_{Bm}/\partial\phi_o \tag{46}$$

While at the mth bottom reflection, expression (44) is up-dated by finding:

1. ϕ_m from equation (40).

2. $\partial\phi_m/\partial\theta_o$ and $\partial\phi_m/\partial\phi_o$. (method not yet shown)

3. $\partial X'_{Bm}/\partial\theta_o$, $\partial Y'_{Bm}/\partial\theta_o$, $\partial X'_{Bm}/\partial\phi_o$, and $\partial Y'_{Bm}/\partial\phi_o$ from expression (28).

When the ray leaves the mth bottom reflection, $\theta_n = \theta'_{Bm}$ for an instant, where θ'_{Bm} is found from equation (39). After reflection, spreading loss terms at desired points on the ray are obtained from expression (42) and up-dated expression (44).

Expression (44) requires the evaluation of ΔS_n, previously defined as the horizontal distance between the mth bottom reflection and point P_n. Let $\overline{\Delta S}_{i-1}$ be the horizontal distance covered as the ray travels between two successive depths, Z_{i-1} and Z_i. Since Z_{Bm} and Z_n are the respective depths of the mth bottom reflection and point P_n,

$$\Delta S_n = \sum_{i=1+Bm}^{n} \overline{\Delta S}_{i-1} \tag{47}$$

In order to derive $\overline{\Delta S}_{i-1}$, recall that the ray path is refracted according to Snell's Law:

$V_a/\cos\theta_a = V_b/\cos\theta_b$ between successive boundary reflections

where V_a, V_b, θ_a, and θ_b are the respective sound velocities and angles of inclination of the ray at any two depths, Z_a and Z_b. Since the ocean surface is assumed to be a horizontal plane, the ray's entrant and emergent angles of inclination at a surface reflection will be of equal magnitude. Therefore, the preceding equation can be extended to read

$$V_a/\cos \theta_a = V_b/\cos \theta_b \quad \text{between successive bottom reflections.} \tag{48}$$

Equation (48) invites an expression of velocity as a function of θ and V_r, where V_r is the reversal velocity — that velocity at which the ray reverses vertical direction because of refraction.

$$V = V_r \cos \theta \qquad \text{between successive bottom reflections} \tag{49}$$

where

$$V_r = V_a/\cos \theta_a = V_b/\cos \theta_b.$$

From the first basic assumption, the ocean velocity structure consists of a series of horizontal layers containing constant velocity gradients.

If k_{i-1} is the velocity gradient between depths Z_{i-1} and Z_i,

$$k_{i-1} = (V_i - V_{i-1})/(Z_i - Z_{i-1}) \tag{50}$$

$$Z = Z_{i-1} + (V - V_{i-1})/k_{i-1} \quad \text{between } Z_{i-1} \text{ and } Z_i. \tag{51}$$

Combining equations (49) and (51),

$$Z = Z_{i-1} + V_r (\cos \theta - \cos \theta_{i-1})/k_{i-1} \quad \text{between } Z_{i-1} \text{ and } Z_i. \tag{52}$$

Equation (52) clearly represents the arc of a circle with radius V_r/k_{i-1}. It follows that the horizontal component of the circular arc between depths Z_{i-1} and Z_i is

$$\overline{\Delta S}_{i-1} = V_r (\sin \theta_{i-1} - \sin \theta_i)/k_{i-1} \tag{53}$$

where V_r is constant between successive bottom reflections and k_{i-1} is constant between depths Z_{i-1} and Z_i.

Since each ocean depth, Z_i, has only one corresponding velocity, V_i, a ray will cross Z_i with identical entrant and emergent angles of inclination, θ_i. Reflection from the level ocean surface will result in negative entrant and positive emergent angles of inclination which are of equal magnitude. Therefore, substituting equation (53) into (47), the horizontal distance from the mth bottom reflection to the nth point on a ray is

$$S_n = V_{r_m} \sum_{i=1+B'_m}^{n} (\sin \theta_{i-1} - \sin \theta_i)/k_{i-1} \qquad (54)$$

where: 1. All velocity gradient layer boundary crossings are represented. 2. Surface reflection entrant and emergent angles of inclination are of equal magnitude and opposite algebraic sign. 3. The mth bottom reflection emergent angle of inclination is linked to earlier terminology by $\theta_{B'_m} \equiv \theta'_{Bm}$. 4. k_{i-1} is obtained from equation (50). 5. V_{r_m} (the ray's reversal velocity after the mth bottom reflection) and θ_i are obtained from V_{Bm} (the velocity at reflection) and θ'_{Bm} by applying equation (49) as follows:

$$V_{r_m} = \frac{V_{i-1}}{\cos \theta_{i-1}} = \frac{V_i}{\cos \theta_i} = \frac{V_{Bm}}{\cos \theta'_{Bm}} \qquad (55)$$

All terms in expression (44) are now available except the stated partial derivatives of ΔS_n and ϕ_m. Combining expressions (54) and (55),

$$\Delta S_n = \sum_{i=1+B'_m}^{n} (V_{i-1} \tan \theta_{i-1} - V_i \tan \theta_i)/k_{i-1}$$

$$\frac{\partial \Delta S_n}{\partial \theta_o} = \frac{\partial}{\partial \theta_o}\left[\sum_{i=1+B'_m}^{n} (V_{i-1} \tan \theta_{i-1} - V_i \tan \theta_i)/k_{i-1}\right] . \qquad (56)$$

Since the velocity gradient may change abruptly at a layer boundary and since the ocean surface and the layer boundaries are all horizontal, discontinuities in the calculation of $\partial \Delta S_n / \partial \theta_o$ are avoided by choosing $\partial Z_i / \partial \theta_o = 0$, where Z_i is the depth of the ith point on a ray. Under this condition, equation (56) can be rewritten:

$$\frac{\partial \Delta S_n}{\partial \theta_o} = \sum_{i=1+B'_m}^{n} \frac{\partial}{\partial \theta_o}\left(\frac{V_{i-1} \tan \theta_{i-1} - V_i \tan \theta_i}{k_{i-1}}\right) . \qquad (57)$$

Since velocity is constant at each depth,

$$\partial V_{i-1}/\partial \theta_o = 0, \quad \partial V_i/\partial \theta_o = 0, \quad \partial V_{Bm}/\partial \theta_o = 0$$

when $\partial Z_{i-1}/\partial \theta_o = 0, \quad \partial Z_i/\partial \theta_o = 0, \quad \partial Z_{Bm}/\partial \theta_o = 0$ (58)

Employing expressions (50) and (58),

$$\partial k_{i-1}/\partial \theta_o = 0 \qquad (59)$$

Therefore, from expressions (55), (57), (58), and (59),

$$\frac{\partial \Delta S_n}{\partial \theta_o} = \sum_{i=1+B'_m}^{n} \frac{1}{k_{i-1}} \left(\frac{V_{i-1}}{\cos^2 \theta_{i-1}} \frac{\partial \theta_{i-1}}{\partial \theta_o} - \frac{V_i}{\cos^2 \theta_i} \frac{\partial \theta_i}{\partial \theta_o} \right)$$

$$= V_{r_m} \sum_{i=1+B'_m}^{n} \frac{1}{k_{i-1}} \left(\frac{1}{\cos \theta_{i-1}} \frac{\partial \theta_{i-1}}{\partial \theta_o} - \frac{1}{\cos \theta_i} \frac{\partial \theta_i}{\partial \theta_o} \right). \tag{60}$$

From expression (55),

$$\cos \theta_i = \frac{V_i}{V_{Bm}} \cos \theta'_{Bm} \tag{61}$$

From expressions (58) and (61),

$$- \sin \theta_i \frac{\partial \theta_i}{\partial \theta_o} = \frac{-V_i}{V_{Bm}} \sin \theta'_{Bm} \frac{\partial \theta'_{Bm}}{\partial \theta_o}$$

$$\frac{\partial \theta_i}{\partial \theta_o} = \frac{\tan \theta'_{Bm}}{\tan \theta_i} \frac{\partial \theta'_{Bm}}{\partial \theta_o} . \tag{62}$$

Similarly,

$$\frac{\partial \theta_{i-1}}{\partial \theta_o} = \frac{\tan \theta'_{Bm}}{\tan \theta_{i-1}} \frac{\partial \theta'_{Bm}}{\partial \theta_o} . \tag{63}$$

Substituting equations (62) and (63) into (60),

$$\frac{\partial \Delta S_n}{\partial \theta_o} = V_{r_m} \tan \theta'_{Bm} \frac{\partial \theta'_{Bm}}{\partial \theta_o} \sum_{i=1+B'_m}^{n} \frac{1}{k_{i-1}} \left(\frac{1}{\sin \theta_{i-1}} - \frac{1}{\sin \theta_i} \right)$$

Assuming that the summation increment in equation (54) is available, it is more convenient to rewrite the above as

$$\frac{\partial \Delta S_n}{\partial \theta_o} = -V_{r_m} \tan \theta'_{Bm} \frac{\partial \theta'_{Bm}}{\partial \theta_o} \sum_{i=1+B'_m}^{n} \left(\frac{\sin \theta_{i-1} - \sin \theta_i}{k_{i-1} \sin \theta_{i-1} \sin \theta_i} \right) \tag{64}$$

Deriving $\partial \Delta S_n / \partial \phi_o$ in a manner similar to equations (56) through (64),

$$\frac{\partial \Delta S_n}{\partial \phi_o} = -V_{r_m} \tan \theta'_{Bm} \frac{\partial \theta'_{Bm}}{\partial \phi_o} \sum_{i=1+\delta_m'}^{n} \left(\frac{\sin \theta_{i-1} - \sin \theta_i}{k_{i-1} \sin \theta_{i-1} \sin \theta_i} \right) . \qquad (65)$$

The above equations are suitable for application in expression (44). It now remains to determine the partial derivatives of θ'_{Bm} and ϕ_m in expressions (44), (64), and (65).

Recall that equations (39) and (40) display a relationship between the ray's mth bottom reflection entrant and emergent ray directions, $(\theta_{Bm}, \phi_{m-1})$ and (θ'_{Bm}, ϕ_m), where θ is the ray's angle of inclination and ϕ is the angle between the vertical XZ plane and the vertical plane containing the ray path. From equation (39),

$$\cos \theta'_{Bm} \frac{\partial \theta'_{Bm}}{\partial \theta_o} = \cos \theta_{Bm} \frac{\partial \theta_{Bm}}{\partial \theta_o} - 2C_{m3} \sin \theta_{Bm} \frac{\partial \theta_{Bm}}{\partial \theta_o} + 2 \cos \theta_{Bm} \frac{\partial C_{m3}}{\partial \theta_o}$$

$$\frac{\partial \theta'_{Bm}}{\partial \theta_o} = \frac{\cos \theta_{Bm}}{\cos \theta'_{Bm}} \left[(1 - 2C_{m3} \tan \theta_{Bm}) \frac{\partial \theta_{Bm}}{\partial \theta_o} + 2 \frac{\partial C_{m3}}{\partial \theta_o} \right] \qquad (66)$$

where, from equation (41),

$$\frac{\partial C_{m3}}{\partial \theta_o} = \frac{(\cos \phi_{m-1} \tan \sigma_m - \sin \phi_{m-1} \tan \varkappa_m) \frac{\partial \phi_{m-1}}{\partial \theta_o} - \frac{1}{\cos^2 \theta_{Bm}} \frac{\partial \theta_{Bm}}{\partial \theta_o}}{\tan^2 \varkappa_m + \tan^2 \sigma_m + 1} . \qquad (67)$$

Deriving $\partial \phi_m / \partial \theta_o$ from equation (40),

$$\frac{1}{\cos^2 \phi_m} \frac{\partial \phi_m}{\partial \theta_o} = \frac{\cos \phi_{m-1} \frac{\partial \phi_{m-1}}{\partial \theta_o} - 2 \tan \sigma_m \frac{\partial C_{m3}}{\partial \theta_o}}{\cos \phi_{m-1} - 2C_{m3} \tan \varkappa_m}$$

$$+ \frac{\sin \phi_{m-1} - 2C_{m3} \tan \sigma_m}{(\cos \phi_{m-1} - 2C_{m3} \tan \varkappa_m)^2} \left(\sin \phi_{m-1} \frac{\partial \phi_{m-1}}{\partial \theta_o} + 2 \tan \varkappa_m \frac{\partial C_{m3}}{\partial \theta_o} \right) .$$

Employing equation (38) in the above,

$$\frac{1}{\cos^2 \phi_m} \frac{\partial \phi_m}{\partial \theta_o} = \frac{\cos \theta_{Bm}}{\cos \theta'_{Bm} \cos \phi_m} \left[\cos \phi_{m-1} \frac{\partial \phi_{m-1}}{\partial \theta_o} - 2 \tan \sigma_m \frac{\partial C_{m3}}{\partial \theta_o} \right.$$

$$\left. + \tan \phi_m \left(\sin \phi_{m-1} \frac{\partial \phi_{m-1}}{\partial \theta_o} + 2 \tan \psi_m \frac{\partial C_{m3}}{\partial \theta_o} \right) \right]$$

$$\frac{\partial \phi_m}{\partial \theta_o} = \frac{\cos \theta_{Bm} \cos \phi_m}{\cos \theta'_{Bm}} \left[(\sin \phi_{m-1} \tan \phi_m + \cos \phi_{m-1}) \frac{\partial \phi_{m-1}}{\partial \theta_o} \right.$$

$$\left. + 2(\tan \phi_m \tan \psi_m - \tan \sigma_m) \frac{\partial C_{m3}}{\partial \theta_o} \right] . \tag{68}$$

Equations (66) through (68) describe $\partial \theta'_{Bm}/\partial \theta_o$ and $\partial \phi_m/\partial \theta_o$ in terms of $\partial \theta_{Bm}/\partial \theta_o$ and $\partial \phi_{m-1}/\partial \theta_o$. However, it is desirable to write $\partial \theta_{Bm}/\partial \theta_o$ in terms of $\partial \theta'_{Bm-1}/\partial \theta_o$ so that such an expression can be used with equations (66) through (68) to establish recursive expressions for $\partial \theta'_{Bm}/\partial \theta_o$ and $\partial \phi_m/\partial \theta_o$ in terms of $\partial \theta'_{Bm-1}/\partial \theta_o$ and $\partial \phi_{m-1}/\partial \theta_o$. The desired expression for $\partial \theta_{Bm}/\partial \theta_o$ can be obtained from the following application of equation (48):

$$\cos \theta_{Bm} = \frac{V_{Bm}}{V_{Bm-1}} \cos \theta'_{Bm-1}$$

where θ'_{Bm-1} is the ray's m-1st bottom reflection emergent angle of inclination and θ_{Bm} is the ray's mth bottom reflection entrant angle of inclination. From expression (58) and the above,

$$-\sin \theta_{Bm} \frac{\partial \theta_{Bm}}{\partial \theta_o} = - \frac{V_{Bm}}{V_{Bm-1}} \sin \theta'_{Bm-1} \frac{\partial \theta'_{Bm-1}}{\partial \theta_o}$$

$$\frac{\partial \theta_{Bm}}{\partial \theta_o} = \frac{\tan \theta'_{Bm-1}}{\tan \theta_{Bm}} \frac{\partial \theta'_{Bm-1}}{\partial \theta_o} . \tag{69}$$

Substituting equation (69) into (67),

$$\frac{\partial C_{m3}}{\partial \theta_o} = C_{m4} \frac{\partial \phi_{m-1}}{\partial \theta_o} + C_{m5} \frac{\partial \theta'_{Bm-1}}{\partial \theta_o} \tag{70}$$

where

$$C_{m4} = \frac{\cos \phi_{m-1} \tan \sigma_m - \sin \phi_{m-1} \tan \psi_m}{\tan^2 \psi_m + \tan^2 \sigma_m + 1}$$

$$C_{m5} = \frac{-\tan \theta'_{Bm-1}}{\sin \theta_{Bm} \cos \theta_{Bm} (\tan^2 \psi_m + \tan^2 \sigma_m + 1)} .$$

Noting a similar derivation of $\partial\theta'_{Bm}/\partial\phi_o$ and $\partial\phi_m/\partial\phi_o$, and substituting equations (69) and (70) into (66) and (68), the recursive expressions for the partial derivatives of θ'_{Bm} and ϕ_m are:

$$\frac{\partial\theta'_{Bm}}{\partial\theta_o} = \frac{\cos \theta_{Bm}}{\cos \theta'_{Bm}} \left[C_{m6} \frac{\partial\theta'_{Bm-1}}{\partial\theta_o} + 2 \left(C_{m4} \frac{\partial\phi_{m-1}}{\partial\theta_o} + C_{m5} \frac{\partial\theta'_{Bm-1}}{\partial\theta_o} \right) \right]$$

$$\frac{\partial\phi_m}{\partial\theta_o} = \frac{\cos \theta_{Bm} \cos \phi_m}{\cos \theta'_{Bm}} \left[C_{m7} \frac{\partial\phi_{m-1}}{\partial\theta_o} + 2C_{m8} \left(C_{m4} \frac{\partial\phi_{m-1}}{\partial\theta_o} + C_{m5} \frac{\partial\theta'_{Bm-1}}{\partial\theta_o} \right) \right]$$

$$\frac{\partial\theta'_{Bm}}{\partial\phi_o} = \frac{\cos \theta_{Bm}}{\cos \theta'_{Bm}} \left[C_{m6} \frac{\partial\theta'_{Bm-1}}{\partial\phi_o} + 2 \left(C_{m4} \frac{\partial\phi_{m-1}}{\partial\phi_o} + C_{m5} \frac{\partial\theta'_{Bm-1}}{\partial\phi_c} \right) \right] \quad (71)$$

$$\frac{\partial\phi_m}{\partial\phi_o} = \frac{\cos \theta_{Bm} \cos \phi_m}{\cos \theta'_{Bm}} \left[C_{m7} \frac{\partial\phi_{m-1}}{\partial\phi_o} + 2C_{m8} \left(C_{m4} \frac{\partial\phi_{m-1}}{\partial\phi_o} + C_{m5} \frac{\partial\theta'_{Bm-1}}{\partial\phi_o} \right) \right]$$

where

$$C_{m4} = \frac{\cos \phi_{m-1} \tan \sigma_m - \sin \phi_{m-1} \tan \psi_m}{\tan^2 \psi_m + \tan^2 \sigma_m + 1}$$

$$C_{m5} = \frac{-\tan \theta'_{Bm-1}}{\sin \theta_{Bm} \cos \theta_{Bm} (\tan^2 \psi_m + \tan^2 \sigma_m + 1)}$$

$$C_{m6} = \frac{\tan \theta'_{Bm-1}}{\tan \theta_{Bm}} - 2C_{m3} \tan \theta'_{Bm-1}$$

$$C_{m7} = \sin \phi_{m-1} \tan \phi_m + \cos \phi_{m-1}$$

$$C_{m8} = \tan \phi_m \tan \psi_m - \tan \sigma_m$$

and from equation (41),

$$C_{m3} = \frac{\cos \phi_{m-1} \tan \psi_m + \sin \phi_{m-1} \tan \sigma_m - \tan \theta_{Bm}}{\tan^2 \psi_m + \tan^2 \sigma_m + 1}$$

GENERAL RAY TRACING PROCEDURE

The spreading loss term at each desired point on a ray is obtained from expressions (42), (44), (54), (55), (64), and (65) where, before the first bottom reflection,

$$m = 0 \qquad \phi_m = \phi_o \qquad \partial\phi_m/\partial\phi_o = 1 \qquad \partial\phi_m/\partial\theta_o = 0$$

$$Bm = 0 \qquad \partial X'_{Bm}/\partial\theta_o = 0 \qquad \partial Y'_{Bm}/\partial\theta_o = 0 \qquad \partial X'_{Bm}/\partial\phi_o = 0$$

$$\partial Y'_{Bm}/\partial\phi_o = 0 \qquad \theta'_{Bm} = \theta_o \qquad \partial\theta'_{Bm}/\partial\theta_o = 1 \qquad \partial\theta'_{Bm}/\partial\phi_o = 0$$

$$V_{r_m} = V_o/\cos\theta_o$$

When the ray reaches the mth bottom reflection point, $\theta_n = \theta_{Bm}$,

$$\partial X_n/\partial\theta_o = \partial X_{Bm}/\partial\theta_o \qquad \partial Y_n/\partial\theta_o = \partial Y_{Bm}/\partial\theta_o$$

$$\partial X_n/\partial\phi_o = \partial X_{Bm}/\partial\phi_o \qquad \partial Y_n/\partial\phi_o = \partial Y_{Bm}/\partial\phi_o$$

While at the mth bottom reflection, expressions (44), (54), (64), and (65) are up-dated by finding:

1. θ'_{Bm} and ϕ_m from equations (39) and (40).

2. $V_{r_m} = V_{Bm}/\cos\theta'_{Bm}$ from expression (55).

3. $\partial\theta'_{Bm}/\partial\theta_o$, $\partial\phi_m/\partial\theta_o$, $\partial\theta'_{Bm}/\partial\phi_o$, and $\partial\phi_m/\partial\phi_o$ from expression (71).

4. $\partial X'_{Bm}/\partial\theta_o$, $\partial Y'_{Bm}/\partial\theta_o$, $\partial X'_{Bm}/\partial\phi_o$, and $\partial Y'_{Bm}/\partial\phi_o$ from expression (28).

After the mth bottom reflection, spreading loss terms at desired points on the ray are obtained from equation (42) and up-dated expressions (44), (54), (64), and (65).

SPECIAL EXAMPLES

There are two cases of particular interest which greatly simplify the preceding ray tracing procedure for calculating the spreading loss term.

1. The initial ray path lies on the vertical XZ plane, and the ocean bottom is described such that it is orthogonal to the XZ plane.

2. The ocean bottom is a horizontal plane.

CASE I

The conditions of the first case are $\phi_o = 0$ and $\sigma = 0$. Substituting $\sigma = 0$ into equation (38a),

$$\sin \phi_m = \frac{\cos \theta_{Bm}}{\cos \theta'_{Bm}} \sin \phi_{m-1} . \tag{72}$$

When $\phi_o = 0$, equation (72) yields

$$\phi_m = \phi_{m-1} = \ldots = \phi_1 = \phi_o = 0. \tag{73}$$

Also from equation (72),

$$\cos \phi_m \frac{\partial \phi_m}{\partial \phi_o} = \frac{\cos \theta_{Bm}}{\cos \theta'_{Bm}} \cos \phi_{m-1} \frac{\partial \phi_{m-1}}{\partial \phi_o} + \sin \phi_{m-1} \frac{\partial}{\partial \phi_o} \left(\frac{\cos \theta_{Bm}}{\cos \theta'_{Bm}} \right) .$$

From expression (73) and the above,

$$\left. \frac{\partial \phi_m}{\partial \phi_o} \right|_{\phi_o = 0} = \frac{\cos \theta_{Bm}}{\cos \theta'_{Bm}} \frac{\partial \phi_{m-1}}{\partial \phi_o} . \tag{74}$$

Expressions (72) and (73) can be used to similarly derive

$$\left. \frac{\partial \phi_m}{\partial \theta_o} \right|_{\phi_o = 0} = \frac{\cos \theta_{Bm}}{\cos \theta'_{Bm}} \frac{\partial \phi_{m-1}}{\partial \theta_o} ,$$

Since $\partial \phi_m / \partial \theta_o = 0$ when $m = 0$, the above equation indicates that for any m,

$$\left. \frac{\partial \phi_m}{\partial \theta_o} \right|_{\phi_o = 0} = 0. \tag{75}$$

Since $\partial \phi_m / \partial \phi_o = 1$ when $m = 0$, equation (74) can be written

$$\left.\frac{\partial \phi_m}{\partial \phi_o}\right|_{\phi_o=0} = \frac{\cos \Theta_{Bm}}{\cos \Theta'_{Bm}} \frac{\cos \Theta_{Bm-1}}{\cos \Theta'_{Bm-1}} \cdots \frac{\cos \Theta_{B1}}{\cos \Theta'_{B1}} . \tag{76}$$

Restating equation (55),

$$V_{r_m} = \frac{V_{Bm}}{\cos \Theta'_{Bm}} \tag{77}$$

where V_{r_m} is the reversal velocity after the mth bottom reflection, V_{Bm} is the velocity at the reflection, and Θ'_{Bm} is the ray's mth bottom reflection emergent angle of inclination. Equation (55) was obtained from equation (49), from which a similar and further application yields

$$V_{r_{m-1}} = \frac{V_{Bm-1}}{\cos \Theta'_{Bm-1}} = \frac{V_{Bm}}{\cos \Theta_{Bm}} \tag{78}$$

where Θ_{Bm} is the ray's mth bottom reflection entrant angle of inclination. Combining equations (77) and (78),

$$\frac{\cos \Theta_{Bm}}{\cos \Theta'_{Bm}} = \frac{V_{r_m}}{V_{r_{m-1}}} . \tag{79}$$

Substituting the above into equation (76),

$$\left.\frac{\partial \phi_m}{\partial \phi_o}\right|_{\phi_o=0} = \frac{V_{r_m}}{V_{r_{m-1}}} \frac{V_{r_{m-1}}}{V_{r_{m-2}}} \cdots \frac{V_{r_1}}{V_{r_o}} = \frac{V_{r_m}}{V_{r_o}} \tag{80}$$

Substituting expressions (73), (75), and (80) into (44),

$$\begin{aligned}
\left.\frac{\partial X_n}{\partial \Theta_o}\right|_{\phi_o=0} &= \frac{\partial X'_{Bm}}{\partial \Theta_o} + \frac{\partial \Delta S_n}{\partial \Theta_o} \\
\left.\frac{\partial Y_n}{\partial \Theta_o}\right|_{\phi_o=0} &= \frac{\partial Y'_{Bm}}{\partial \Theta_o} \\
\left.\frac{\partial X_n}{\partial \phi_o}\right|_{\phi_o=0} &= \frac{\partial X'_{Bm}}{\partial \phi_o} + \frac{\partial \Delta S_n}{\partial \phi_o} \\
\left.\frac{\partial Y_n}{\partial \phi_o}\right|_{\phi_o=0} &= \frac{\partial Y'_{Bm}}{\partial \phi_o} + \frac{V_{r_m}}{V_{r_o}} \Delta S_n
\end{aligned} \tag{81}$$

where the primed partial derivatives are found from $\partial X_{Bm}/\partial\theta_o$, $\partial Y_{Bm}/\partial\theta_o$, $\partial X_{Bm}/\partial\phi_o$, and $\partial Y_{Bm}/\partial\phi_o$, using expression (28).

Expression (28) is an ocean bottom boundary condition approximation which can be simplified by employing expression (73) and $\sigma = 0$.

$$\left.\frac{\partial X'_{Bm}}{\partial\theta_o}\right|_{\phi_o=0} = \frac{\tan\theta_{Bm}}{\tan\theta'_{Bm}}\left(\frac{\tan\theta'_{Bm} - \tan\psi_m}{\tan\theta_{Bm} - \tan\psi_m}\right)\frac{\partial X_{Bm}}{\partial\theta_o}$$

$$\left.\frac{\partial Y'_{Bm}}{\partial\theta_o}\right|_{\phi_o=0} = \frac{\partial Y_{Bm}}{\partial\theta_o}$$

$$\left.\frac{\partial X'_{Bm}}{\partial\phi_o}\right|_{\phi_o=0} = \frac{\tan\theta_{Bm}}{\tan\theta'_{Bm}}\left(\frac{\tan\theta'_{Bm} - \tan\psi_m}{\tan\theta_{Bm} - \tan\psi_m}\right)\frac{\partial X_{Bm}}{\partial\phi_o} \qquad (82)$$

$$\left.\frac{\partial Y'_{Bm}}{\partial\phi_o}\right|_{\phi_o=0} = \frac{\partial Y_{Bm}}{\partial\phi_o}$$

Let ΔS_{m-1} be the horizontal distance between the m-1st and mth bottom reflections. Since the respective coordinates of the reflection points are $(X_{Bm-1}, Y_{Bm-1}, Z_{Bm-1})$ and (X_{Bm}, Y_{Bm}, Z_{Bm}), Fig. 13 (page 19) indicates that:

$$X_{Bm} = X_{Bm-1} + \Delta S_{m-1}\cos\phi_{m-1}$$

$$Y_{Bm} = Y_{Bm-1} + \Delta S_{m-1}\sin\phi_{m-1}$$

The boundary condition discussion preceding expression (28) used unprimed and primed partial derivatives to respectively denote before- and after-reflection rates of change. Bearing this in mind, the above equations are used to obtain

$$\frac{\partial X_{Bm}}{\partial\theta_o} = \frac{\partial X'_{Bm-1}}{\partial\theta_o} - \Delta S_{m-1}\sin\phi_{m-1}\frac{\partial\phi_{m-1}}{\partial\theta_o} + \cos\phi_{m-1}\frac{\partial\Delta S_{m-1}}{\partial\theta_o}$$

$$\frac{\partial Y_{Bm}}{\partial\theta_o} = \frac{\partial Y'_{Bm-1}}{\partial\theta_o} + \Delta S_{m-1}\cos\phi_{m-1}\frac{\partial\phi_{m-1}}{\partial\theta_o} + \sin\phi_{m-1}\frac{\partial\Delta S_{m-1}}{\partial\theta_o}$$

$$\frac{\partial Y_{Bm}}{\partial\phi_o} = \frac{\partial Y'_{Bm-1}}{\partial\phi_o} + \Delta S_{m-1}\cos\phi_{m-1}\frac{\partial\phi_{m-1}}{\partial\phi_o} + \sin\phi_{m-1}\frac{\partial\Delta S_{m-1}}{\partial\phi_o}.$$

From expressions (73), (75), (80), and the previous equation,

$$\left.\frac{\partial X_{Bm}}{\partial \theta_o}\right|_{\phi_o=0} = \frac{\partial X'_{Bm-1}}{\partial \theta_o} + \frac{\partial \Delta S_{m-1}}{\partial \theta_o}$$

$$\left.\frac{\partial Y_{Bm}}{\partial \theta_o}\right|_{\phi_o=0} = \frac{\partial Y'_{Bm-1}}{\partial \theta_o} \qquad (83)$$

$$\left.\frac{\partial Y_{Bm}}{\partial \phi_o}\right|_{\phi_o=0} = \frac{\partial Y'_{Bm-1}}{\partial \phi_o} + \frac{V_{r_{m-1}}}{V_{r_o}} \Delta S_{m-1} .$$

Combining expressions (82) and (83),

$$\left.\frac{\partial X'_{Bm}}{\partial \theta_o}\right|_{\phi_o=0} = \frac{\tan \theta_{Bm}}{\tan \theta'_{Bm}} \left(\frac{\tan \theta'_{Bm} - \tan \psi_m}{\tan \theta_{Bm} - \tan \psi_m}\right)\left(\frac{\partial X'_{Bm-1}}{\partial \theta_o} + \frac{\partial \Delta S_{m-1}}{\partial \theta_o}\right)$$

$$\left.\frac{\partial Y'_{Bm}}{\partial \theta_o}\right|_{\phi_o=0} = \frac{\partial Y'_{Bm-1}}{\partial \theta_o}$$

$$\left.\frac{\partial Y'_{Bm}}{\partial \phi_o}\right|_{\phi_o=0} = \frac{\partial Y'_{Bm-1}}{\partial \phi_o} + \frac{V_{r_{m-1}}}{V_{r_o}} \Delta S_{m-1} .$$

Since $\partial X'_{Bm}/\partial \theta_o$, $\partial Y'_{Bm}/\partial \theta_o$, and $\partial Y'_{Bm}/\partial \phi_o = 0$ when $m = 0$, the above expression indicates that for any $m > 0$,

$$\left.\frac{\partial X'_{Bm}}{\partial \theta_o}\right|_{\phi_o=0} = C_m \left(\frac{\partial X'_{Bm-1}}{\partial \theta_o} + \frac{\partial \Delta S_{m-1}}{\partial \theta_o}\right) \qquad (84)$$

where $\partial X'_{Bm-1}/\partial \theta_o = 0$ when $m = 1$

and $C_m = \dfrac{\tan \theta_{Bm}}{\tan \theta'_{Bm}} \left(\dfrac{\tan \theta'_{Bm} - \tan \psi_m}{\tan \theta_{Bm} - \tan \psi_m}\right)$

$$\left.\frac{\partial Y'_{Bm}}{\partial \theta_o}\right|_{\phi_o=0} = 0$$

$$\left.\frac{\partial Y'_{Bm}}{\partial \phi_o}\right|_{\phi_o=0} = \frac{1}{V_{r_o}} \sum_{j=0}^{m-1} V_{r_j} \Delta S_j$$

From expression (81) and above,

$$\left.\frac{\partial Y_n}{\partial \theta_o}\right|_{\phi_o=0} = 0 \tag{85}$$

$$\left.\frac{\partial Y_n}{\partial \phi_o}\right|_{\phi_o=0} = \frac{1}{V_{r_o}}\left(\sum_{j=0}^{m-1} V_{r_j} \Delta S_j + V_{r_m} \Delta S_n\right)$$

$$= \frac{\cos\theta_o}{V_o}\left(\sum_{j=0}^{m-1} V_{r_j} \Delta S_j + V_{r_m} \Delta S_n\right) \tag{85a}$$

Combining equations (85), (85a), and (42), when $\phi_o = 0$ and $\sigma = 0$, the special expression for the spreading loss term at point P_n on a ray after m bottom reflections is

$$(SL)_n = \left|\frac{\cos\theta_o}{\sin\theta_n}\left(\frac{\partial X_n}{\partial \theta_o}\frac{\partial Y_n}{\partial \phi_o}\right)^{-1}\right|$$

$$= \left|\frac{\cos\theta_o}{\sin\theta_n}\left[\frac{\partial X_n}{\partial \theta_o}\frac{\cos\theta_o}{V_o}\left(\sum_{j=0}^{m-1} V_{r_j} \Delta S_j + V_{r_m} \Delta S_n\right)\right]^{-1}\right|$$

$$= \left|\frac{V_o}{\sin\theta_n}\left[\frac{\partial X_n}{\partial \theta_o}\left(\sum_{j=0}^{m-1} V_{r_j} \Delta S_j + V_{r_m} \Delta S_n\right)\right]^{-1}\right| \tag{86}$$

where, from expression (81),

$\left.\frac{\partial X_n}{\partial \theta_o}\right|_{\phi_o=0} = \frac{\partial X'_{Bm}}{\partial \theta_o} + \frac{\partial \Delta S_n}{\partial \theta_o}$ with $\partial X'_{Bm}/\partial\theta_o$ described in expression (84).

Pertinent terms in expressions (86) and (84) are defined as follows:

V_o is the velocity at the ray source.

V_{r_j} is the reversal velocity between the jth and j+1st bottom reflections.

ΔS_j is the horizontal distance between the jth and j+1st bottom reflections.

ΔS_n is the horizontal distance from the mth bottom reflection to point P_n.

C_m is the mth bottom reflection boundary condition correction.

θ_{Bm} and θ'_{Bm} are the ray's respective mth bottom reflection entrant and emergent angles of inclination.

ψ_m is the angle the ocean bottom tangent plane at the mth reflection makes with the horizontal in the XZ plane.

It is interesting to note that $C_m = 1$ if the boundary condition is completely ignored, as was done in reference (2). Under this condition, expression (84) yields

$$\left.\frac{\partial X'_{Bm}}{\partial \theta_o}\right|_{\phi_o=0} = \sum_{j=0}^{m-1} \frac{\partial \Delta S_j}{\partial \theta_o}$$

which is substituted into expression (86) to obtain

$$(SL)_n^{-1} = \left| \sin \theta_n \left(\sum_{j=0}^{m-1} \frac{\partial \Delta S_j}{\partial \theta_o} + \frac{\partial \Delta S_n}{\partial \theta_o} \right) \left(\sum_{j=0}^{m-1} \frac{V_{r_j}}{V_o} \Delta S_j + \frac{V_{r_m}}{V_o} \Delta S_n \right) \right|$$

Since $(SL)_n^{-1} + I_o/I_n$ from equation (1), it can be seen that the above equation agrees with equation (1) of reference (2).

Expression (86) is written in terms of ΔS_j, ΔS_n, and $\partial \Delta S_n/\partial \theta_o$, where ΔS_n is found from equation (54). Similarly,

$$\Delta S_j + V_{r_j} \sum_{i=1+B_j}^{B_{j+1}} (\sin \theta_{i-1} - \sin \theta_i)/k_{i-1} \tag{87}$$

$\partial \Delta S_n / \partial \theta_o$ is obtained from equation (64), where $\partial \theta'_{Bm} / \partial \theta_o$, in turn, is described in expression (71). However, when $\phi_o = 0$ and $\sigma = 0$, a much simpler expression for $\partial \theta'_{Bm} / \partial \theta_o$ can be found by first substituting $\sigma = 0$ into equation (41):

$$C_{m3} = (\cos \phi_{m-1} \tan \psi_m - \tan \theta_{Bm}) \cos^2 \psi_m \quad .$$

Employing expression (73),

$$C_{m3} = (\tan \psi_m - \tan \theta_{Bm}) \cos^2 \psi_m \qquad \text{when } \phi_o = 0 \quad .$$

Substituting the above into equation (39), where θ_{Bm} and θ'_{Bm} are the ray's respective mth bottom reflection entrant and emergent angles of inclination,

$$\sin \theta'_{Bm} = \sin \theta_{Bm} + 2 \sin \psi_m \cos \psi_m \cos \theta_{Bm} - 2 \cos^2 \psi_m \sin \theta_{Bm}$$

$$= \sin (2 \psi_m - \theta_{Bm})$$

$$\theta'_{Bm} = 2 \psi_m - \theta_{Bm} \qquad \text{when } \phi_o = 0 \quad . \tag{88}$$

From equations (88) and (69),

$$\left. \frac{\partial \theta'_{Bm}}{\partial \theta_o} \right|_{\phi_o = 0} = - \frac{\partial \theta_{Bm}}{\partial \theta_o} = - \frac{\tan \theta'_{Bm-1}}{\tan \theta_{Bm}} \frac{\partial \theta'_{Bm-1}}{\partial \theta_o} \quad . \tag{89}$$

Noting that

$$\theta'_{Bm} = \theta_o \qquad \text{and} \qquad \partial \theta'_{Bm} / \partial \theta_o = 1 \qquad \text{when } m = 0, \tag{90}$$

equation (89) indicates that for any $m > 0$,

$$\left. \frac{\theta'_{Bm}}{\theta_o} \right|_{\phi_o = 0} = \prod_{j=0}^{m-1} - \frac{\tan \theta'_{Bj}}{\tan \theta_{B_{j+1}}} = \frac{\tan \theta_o}{\tan \theta'_{Bm}} \left(\prod_{j=1}^{m} - \frac{\tan \theta'_{B_j}}{\tan \theta_{B_j}} \right) \tag{91}$$

Substituting equations (90) and (91) into (64),

$$\left.\frac{\partial \Delta S_n}{\partial \theta_o}\right|_{\phi_o=0} = - V_{r_m} \tan \theta_o \left(\prod_{j=1}^{m} - \frac{\tan \theta'_{B_j}}{\tan \theta_{B_j}} \right) \sum_{i=1+\theta_m}^{n} \left(\frac{\sin \theta_{i-1} - \sin \theta_i}{k_{i-1} \sin \theta_{i-1} \sin \theta_i} \right) \tag{92}$$

where the continued product equals one when m = 0.

When $\phi_o = 0$ and $\sigma = 0$, the spreading loss term at each desired point on a ray after m bottom reflections is obtained from expressions (86), (54), and (92). Just as the ray reached the mth reflection,

$$\theta_n = \theta_{Bm} \qquad \Delta S_n = \Delta S_{m-1} \qquad \partial \Delta S_n / \partial \theta_o = \partial \Delta S_{m-1} / \partial \theta_o$$

with expressions (86), (54), and (92) up-dated through reflection by finding:

1. $\theta'_{Bm} = 2 \psi_m - \theta_{Bm}$ from equation (88).
2. $V_{r_m} = V_{Bm} / \cos \theta'_{Bm}$ from equation (77).
3. $\partial X'_{Bm} / \partial \theta_o$ from expression (84).

The validity of spreading loss expression (86) is partly substantiated by its agreement with equation 3B-42 of reference (1) when no bottom reflections are involved. Under this condition, m = 0 and $\partial X'_{Bm} / \partial \theta_o = 0$, which are used in expression (86) to obtain

$$(SL)_n = \left| \frac{V_o}{\sin \theta_n} \left(\frac{\partial \Delta S_n}{\partial \theta_o} \cdot V_{r_o} \Delta S_n \right)^{-1} \right|$$

$$= \left| \frac{\cos \theta_o}{\sin \theta_n} \left(\Delta S_n \frac{\partial \Delta S_n}{\partial \theta_o} \right)^{-1} \right| \quad . \tag{93}$$

Let S_n be the horizontal distance from ray source to point P_n. Then before the first bottom reflection, $S_n = \Delta S_n$, previously defined as the horizontal distance from the mth bottom reflection to point P_n. Therefore, from equations (1) and (93), the spreading loss term at point P_n on a ray before any bottom reflections is

$$(SL)_n^{-1} = \frac{I_o}{I_n} = \left| \frac{\sin \theta_n}{\cos \theta_o} S_n \frac{\partial S_n}{\partial \theta_o} \right| \tag{94}$$

which agrees with equation 3B-42 of reference (1).

CASE II

The ocean bottom is a horizontal plane when its tangent plane inclination angles, ψ and σ, are zero. A ray path leaves its source with direction (θ_o, ϕ_o), where θ_o is the ray's initial angle of inclination and ϕ_o is the angle between the vertical XZ plane and the vertical plane containing the ray path before any bottom reflection. When $\psi = 0$ and $\sigma = 0$, the environment is everywhere symmetric about the vertical Z axis passing through the ray source. Therefore, the family of ray paths with a common source and a common initial angle of inclination will have identical physical characteristics at a given horizontal range, regardless of the value of ϕ_o. The simplest representative ray to study is that in which $\phi_o = 0$.

Since the conditions of Case I were $\sigma = 0$ and $\phi_o = 0$ and the conditions of this horizontal bottom case are $\psi = 0$, $\sigma = 0$, and chosen $\phi_o = 0$, Case I spreading loss expression (86) can be used to derive the spreading loss expression for this case. Setting $\psi = 0$ in expression (84),

$$C_m = 1 \qquad \left.\frac{X_{Bm}}{\theta_o}\right|_{\phi_o=0} = \sum_{j=0}^{m-1} \frac{\partial \Delta S_j}{\partial \theta_o}$$

which is substituted into expression (86) to obtain

$$\left.\frac{\partial X_n}{\partial \theta_o}\right|_{\phi_o=0} = \sum_{j=0}^{m-1} \frac{\partial \Delta S_j}{\partial \theta_o} + \frac{\partial \Delta S_n}{\partial \theta_o} \quad . \tag{95}$$

If S_n is the horizontal distance from ray source to point P_n,

$$S_n = \sum_{j=0}^{m-1} \Delta S_j + \Delta S_n \tag{96}$$

$$\frac{\partial S_n}{\partial \theta_o} = \frac{\partial}{\partial \theta_o}\left(\sum_{j=0}^{m-1} \Delta S_j + \Delta S_n \right) \quad . \tag{97}$$

It has been consistently assumed that $\partial Z_i / \partial \theta_o = 0$, where Z_i is the depth at any given point P_i on a ray. Therefore, since reflection points on the horizontal ocean bottom satisfy the above condition, equation (97) can be rewritten

$$\frac{\partial S_n}{\partial \theta_o} = \sum_{j=0}^{m-1} \frac{\partial \Delta S_j}{\partial \theta_o} + \frac{\partial \Delta S_n}{\partial \theta_o} \tag{97a}$$

When the ocean bottom is horizontal, spreading loss expression (86) is simplified by noting the equality of equation (95) and the above.

$$\left.\frac{\partial X_n}{\partial \theta_o}\right|_{\phi_o=0} = \frac{\partial S_n}{\partial \theta_o} \quad . \tag{98}$$

A horizontal bottom also simplifies the relationship between entrant and emergent inclination angles of a ray at reflection. Setting $\psi = 0$ in equation (88),

$$\theta'_{Bm} = -\theta_{Bm} \quad . \tag{99}$$

Employing the above in equation (79),

$$V_{r_m}/V_{r_{m-1}} = 1 \quad .$$

It follows that

$$V_{r_m} = V_{r_{m-1}} = \dots = V_{r_j} = \dots = V_{r_o} \quad . \tag{100}$$

Equations (96) and (100) indicate that

$$\sum_{j=0}^{m-1} V_{r_j} \Delta S_j + V_{r_m} \Delta S_n = V_{r_o} \left(\sum_{j=0}^{m-1} \Delta S_j + \Delta S_n \right)$$

$$= V_{r_o} S_n = \frac{V_o S_n}{\cos \theta_o} \quad .$$

Substituting equation (98) and the above into expression (86), when the ocean bottom is a horizontal plane, the special expression for the spreading loss term at point P_n anywhere on a ray is

$$(SL)_n = \left| \frac{V_o}{\sin \theta_n} \left(\frac{\partial S_n}{\partial \theta_o} \frac{V_o \, S_n}{\cos \theta_o} \right)^{-1} \right|$$

$$= \left| \frac{\cos \theta_o}{\sin \theta_n} \left(S_n \frac{\partial S_n}{\partial \theta_o} \right)^{-1} \right| \tag{101}$$

which happens to agree with equation (94).

An evaluation of S_n is obtained from equations (96), (87), and (54).

$$S_n = \sum_{j=0}^{m-1} V_{r_j} \left(\sum_{i=1+B_j'}^{B_j+1} (\sin \theta_{i-1} - \sin \theta_i)/k_{i-1} \right) + V_{r_m} \sum_{i=1+B_m'}^{n} (\sin \theta_{i-1} - \sin \theta_i)/k_{i-1}$$

Employing equations (99) and (100) in the above,

$$S_n = V_{r_o} \sum_{i=1}^{n} (\sin \theta_{i-1} - \sin \theta_i)/k_{i-1} \tag{102}$$

where entrant and emergent angles of inclination at reflections are of equal magnitude and opposite sign. A similar handling of equations (97a), (92), (99), and (100) produces

$$\frac{\partial S_n}{\partial \theta_o} = - V_{r_o} \tan \theta_o \sum_{i=1}^{n} \left(\frac{\sin \theta_{i-1} - \sin \theta_i}{k_{i-1} \sin \theta_{i-1} \sin \theta_i} \right) . \tag{103}$$

Equations (101), (102), and (103) provide a detailed expression for the spreading loss term at the nth point on a ray in an environment containing a horizontal ocean bottom.

CONCLUSION

The general ray tracing procedure on page 27, the special ray tracing procedure on page 35, and equations (101), (102), and (103) provide spreading loss descriptions which are developed for use with different ocean bottom requirements. Each approach is designed for efficient application on a high-speed digital computer.

Appendix

SPECIAL APPLICATION OF THE AREA OF A QUADRILATERAL

THEOREM

The area of a quadrilateral is equal to eight times the area of any triangle whose vertices are the midpoints of two adjacent sides of the quadrilateral and the point of intersection of the two lines joining opposite midpoints.

PROOF

Construct quadrilateral ABCD with diagonal d between points B and D, and with side midpoints E, F, G, and H as shown in Fig. 1A.

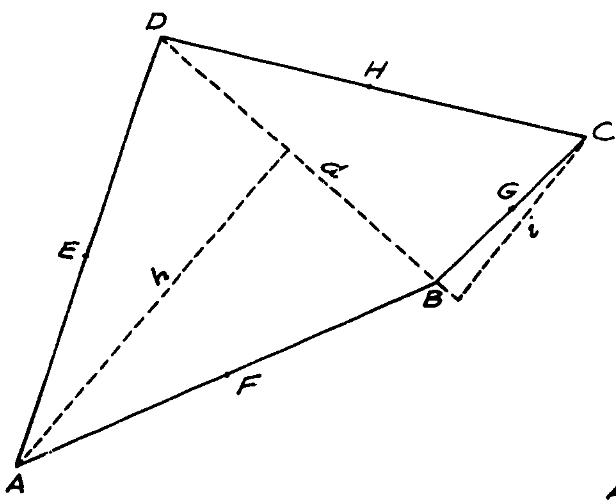

FIG. 1A

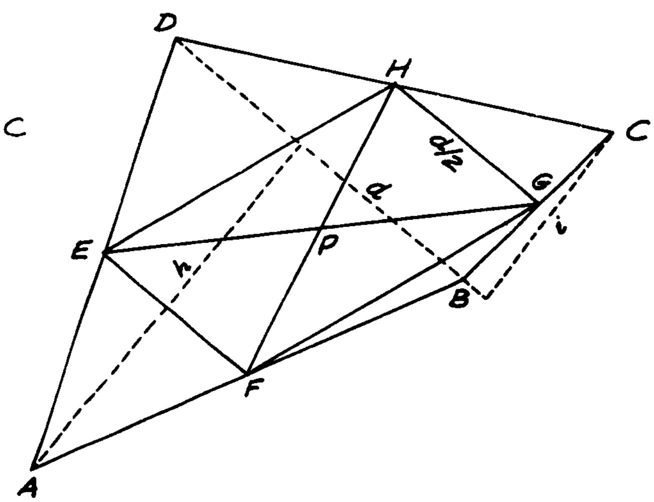

FIG. 2A

Drop perpendiculars h and i from respective points A and C to diagonal d. From Fig. 1A,

$$\text{Area of quadrilateral ABCD} = hd/2 + id/2 = (h + i)\ d/2. \qquad (1)$$

Construct lines joining side midpoints E, F, G, and H as shown in Fig. 2A. From triangle similarities (examples: GCH and BCD, EAF and DAB),

$$\text{EFGH is a parallelogram with base } d/2 \text{ and altitude } (h/2 + i/2). \qquad (2)$$

Construct the diagonals of parallelogram EFGH with their intersection

at P.

Theorem: The diagonals of a parallelogram bisect each other. (3)

From expressions (2) and (3), triangles FPE and HPG are congruent, with base d/2 and altitude (h + i)/4. Therefore,

Area of triangle FPE = (h + i)d/16. (4)

Combining equations (1) and (4),

Area of quadrilateral ABCD equals 8 times area of triangle FPE. (5)

By dropping a perpendicular from point E to diagonal FH, it is seen that the areas of triangles FPE and EPH are equal, since they have equal bases and a common altitude. It can be similarly shown that

Triangles FPE, EPH, HPG, and GPF all have equal areas. (6)

Expressions (5) and (6) complete the proof.

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2. E. S. Eby and L. T. Einstein, Spreading Loss for a Special Case of Multiple Bottom Reflections, U. S. L. Technical Memorandum No. 1210-104-60.

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